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ON GREEN'S FUNCTIONS AND SAINT-VENANT'S PRINCIPLE IN THE LINEAR THEORY OF VISCOELASTICITY

by

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Contents

Introduction 1

1. Notation. Preliminaries 4

2. Kelvin's problem in viscoelasticity theory: the basic singular state. Higher-order singular states 16

3. Green's states. Integral representations for the solution of the fundamental boundary-value problems . . 31

4. Saint-Venant's principle for viscoelastic solids 48

References 68

Introduction

In a previous paper [1][1] an attempt was made to contribute toward a comprehensive and — hopefully — rigorous analytical theory of linear viscoelastic behavior, comparable in scope and character to the systematic body of general propositions that has long been available in the classical theory of elastic solids. The material contained in [1], which is confined to an isothermal quasi-static treatment of the subject, includes: results concerning the structure of the relevant constitutive relations and the connection between various alternative versions of the stress-strain law; conclusions regarding the nature of the time and position dependence of solutions to the governing field equations; integral, reciprocal, and uniqueness theorems; and theorems pertaining to the integration of the field equations in terms of complete systems of stress functions. Later on, Gurtin [2] obtained generalizations to viscoelasticity theory of the known variational principles of elastostatics. Also, certain results appearing in [1] have since been extended to non-isothermal conditions in [3],[4].

In the present paper we continue the project begun in [1] and turn to the study of Green's functions in the isothermal quasi-static theory of a homogeneous and isotropic viscoelastic medium which obeys the general linear relaxation integral law.

[1] Numbers in brackets refer to the list of publications at the end of this paper.

Our main objective in this connection is twofold: first, we aim at integral representations for the solution of the standard boundary-value problems, analogous to those originated in elasticity theory by[2] Betti, Volterra, Lauricella, and Somigliana; second, we seek to apply such integral representations to a proof of Saint-Venant's principle in a formulation that is the counterpart in viscoelasticity of the elastostatic principle suggested by von Mises [6] and established in [7].

The chief mathematical tool in this investigation is once again supplied by the algebra and calculus of Stieltjes convolutions developed in [1]. Further, some of the results on linear viscoelasticity appearing in [1] are essential prerequisites to our current purpose. For this reason, and in order to render the present paper sensibly self-contained, we collect in Section 1 various definitions and theorems, mostly adapted from [1], which are needed repeatedly in the subsequent analysis.

In Section 2 we deal first with the problem of a concentrated load acting at a point of a viscoelastic solid that occupies the entire space. The solution to this singular problem, which is a generalization of Kelvin's problem in elastostatics, is defined and deduced explicitly by means of a limit process applied to the solution of a sequence of regular problems governed by distributed body forces. The basic singular solution of the

[2] See Love [5], Art. 169, for detailed references.

field equations thus established is subsequently used to generate the higher-order singularities appropriate to force doublets with and without moment. Certain relevant properties of the singular solutions arrived at in this section are studied in detail.

The results obtained in Section 2 are applied in Section 3 to the contruction of Green's functions and the derivation of integral representations for the solution to the fundamental boundary-value problems in linear viscoelasticity theory. Both Section 2 and Section 3 are influenced by a partly parallel treatment in [8] of the analogous topics in the classical equilibrium theory of elastic solids.

The integral representations obtained in Section 3 are, in turn, employed in Section 4 to prove a Saint-Venant principle appropriate to viscoelastic solids within the theoretical framework underlying this paper.

1. Notation. Preliminaries.

In this section we cite — occasionally in a modified or extended form adapted to our present needs — certain definitions and results appearing in [1] that will be required repeatedly later on.

Throughout this paper the letter E designates a three-dimensional Euclidean space. The symbol $\mathfrak{R}$, in the absence of any qualifying restrictions, will denote an arbitrary region in E that may be either open or closed. Further, we employ the standard notation for closure; thus, if $\mathfrak{R}$ is open, $\bar{\mathfrak{R}}$ stands for the closure of $\mathfrak{R}$. On the other hand, the letter R is consistently reserved for a regular region in E, by which we mean an open (not necessarily bounded or simply connected) region, the boundary B of which consists of a finite number of non-intersecting closed regular surfaces, the latter term being used in the sense of Kellogg [9](p.112). Since B is bounded even if R is not, an unbounded R is of necessity an exterior region, i.e. a region containing all sufficiently distant points. Note also that B may have corners and edges. A point P of B will be referred to as a regular boundary point if B possesses a tangent plane at P; by a regular subset of B we shall mean one that consists exclusively of regular boundary points.

In the present investigation we have frequent occasion to deal with real-valued functions of position and time, whose domain of definition is the cartesian product of a region of

space $\mathcal{R}$ and an (open, closed, or half-open) interval of time, for which we use the symbol $\mathcal{J}$. In this physical context we shall denote by $\underline{x}$, with the rectangular coordinates (x_1, x_2, x_3), the position vector of points in $\mathcal{R}$, call t the time, and write $\mathcal{R}\times\mathcal{J}$ for the appropriate domain of definition. Further, if f is a function defined on $\mathcal{R}\times\mathcal{J}$, we write $f(\underline{x},t)$ for the value of f at $(\underline{x},t)$ and use $f(\cdot,t)$ to designate the subsidiary mapping of position that results from holding t fixed in $\mathcal{J}$. The analogcus interpretation applies to $f(\underline{x},\cdot)$. Finally, as far as the partial space and time differentiation of f is concerned, we adopt the notation

$$\left.\begin{aligned} f^{(n)}_{\underbrace{,ij\ldots\ldots k}_{\text{m indices}}}(\underline{x},t) &\equiv \frac{\partial^{m+n} f(\underline{x},t)}{\partial x_i \partial x_j \ldots\ldots \partial x_k \partial t^n} \quad (m,n = 0,1,2,\cdots\cdot)\ , \\ \dot{f}_{\underbrace{,ij\ldots\ldots k}_{\text{m indices}}}(\underline{x},t) &\equiv \frac{\partial^{m+1} f(\underline{x},t)}{\partial x_i \partial x_j \ldots\ldots \partial x_k \partial t} \quad (m = 0,1,2,\cdots\cdot)\ , \end{aligned}\right\} \quad (1.1)$$

with the understanding that all subscripts, unless otherwise specified, henceforth have the range of the integers (1,2,3).

Turning to functions of the time alone, we define the <u>Heaviside unit step function</u> by

$$\left.\begin{aligned} &(a)\quad h(t) = 0 \quad \text{for} \quad t\in(-\infty,0), \\ &(b)\quad h(t) = 1 \quad \text{for} \quad t\in[0,\infty)\ , \end{aligned}\right\} \quad (1.2)$$

and introduce the following convenient generalization of the Heaviside function.

Definition 1.1 (Functions in Heaviside Class H^N). We say that $f \in H^N$ if $f(t)$ is defined for all $t \in (-\infty,\infty)$ and

$$\left.\begin{array}{ll}\text{(a)} & f = 0 \ \text{on} \ (-\infty,0),\\ \text{(b)} & f \in C^N([0,\infty)).\end{array}\right\} \quad (1.3)^3$$

Further, if $f \in H^N$, we write

$$f^{(n)}(0) \equiv f^{(n)}(0+) \ (n = 0,1,2,\ldots N). \quad (1.4)$$

A useful extension of this definition to functions of position and time is furnished by

Definition 1.2 (Functions in Class $C^{M,N}$ or Class $H^{M,N}$).

(A) We say that $f \in C^{M,N}(\mathcal{R}\times\mathcal{J})$ if $f(\underline{x},t)$ is defined for all $(\underline{x},t) \in \mathcal{R}\times\mathcal{J}$ and the functions

$$f^{(n)}_{,\underbrace{ij\ldots\ldots\ldots k}_{m \text{ indices}}} \quad (m = 0,1,2,\cdots M;\ n = 0,1,2,\cdots N) \quad (1.5)$$

exist and are continuous on $\mathcal{R}\times\mathcal{J}$.

(B) We say that $f \in H^{M,N}(\mathcal{R})$ if $f(\underline{x},t)$ is defined for all $(\underline{x},t) \in \mathcal{R}\times(-\infty,\infty)$ and

$$\left.\begin{array}{ll}\text{(a)} & f = 0 \ \text{on} \ \mathcal{R}\times(-\infty,0),\\ \text{(b)} & f \in C^{M,N}(\mathcal{R}\times[0,\infty)).\end{array}\right\} \quad (1.6)$$

Further, if $f \in H^{M,N}(\mathcal{R})$, we write

$$f^{(n)}_{,\underbrace{ij\ldots\ldots\ldots k}_{m \text{ indices}}}(\cdot,0) \equiv f^{(n)}_{,\underbrace{ij\ldots\ldots\ldots k}_{m \text{ indices}}}(\cdot,0+) \ (m = 0,1,2,\cdots M;\ n = 0,1,2,\cdots N). \quad (1.7)$$

---

[3] If X is a set then $C^N(X)$ stands for the set of all functions that are N times continuously differentiable on X.

Next we collect some basic results from the theory of Stieltjes convolutions. To this end suppose that φ and ψ are functions of position and time defined on $\mathcal{R}\times[0,\infty)$ and $\mathcal{R}\times(-\infty,\infty)$, respectively, and assume that the Riemann-Stieltjes integral

$$\vartheta(\underline{x},t) = \int_{\tau=-\infty}^{t} \varphi(\underline{x},t-\tau)d\psi(\underline{x},\tau) \tag{1.8}$$

exists for all $(\underline{x},t)\in\mathcal{R}\times(-\infty,\infty)$. Then the function ϑ so defined on $\mathcal{R}\times(-\infty,\infty)$ is said to be the <u>Stieltjes</u> <u>convolution</u> of φ and ψ; we also write

$$\vartheta = \varphi*d\psi\ ,\ \vartheta(\underline{x},t) = [\varphi*d\psi](\underline{x},t). \tag{1.9}$$

In view of Theorem 1.2 and Theorem 1.6 of [1], we have

<u>Theorem</u> 1.1 (Properties of Stieltjes convolutions). <u>Let</u> $\varphi\in H^{M,1}(\mathcal{R})$ <u>and</u> $\psi,\theta\in H^{N,1}(\mathcal{R})$. <u>Then</u>:

(a) $\varphi*d\psi\in H^{K,1}(\mathcal{R})$ <u>with</u> $K = \min(M,N)$;

(b) $\varphi*d\psi = \psi*d\varphi$;

(c) $\varphi*d(\psi*d\theta) = (\varphi*d\psi)*d\theta = \varphi*d\psi*d\theta$;

(d) $\varphi*d(\psi+\theta) = \varphi*d\psi + \varphi*d\theta$;

(e) $\varphi*d\psi = 0 \Longrightarrow \varphi = 0$ <u>or</u> $\psi = 0$;

(f) $\varphi*dh = \varphi$;

(g) $[\varphi*d\psi](\underline{x},t) = \psi(\underline{x},0)\varphi(\underline{x},t) + \int_0^t \varphi(\underline{x},t-\tau)\dot{\psi}(\tau)d\tau$ <u>for</u> <u>all</u> $(\underline{x},t)\in\mathcal{R}\times[0,\infty)$;

(h) $(\psi*d\theta)_{,i} = \psi_{,i}*d\theta + \psi*d\theta_{,i}$ <u>if</u> $N \geq 1$.

Theorem 1.3 and Theorem 1.4 of [1] may be combined into

<u>Theorem</u> 1.2 (Stieltjes inverse). <u>Let</u> $\varphi\in H^2$ <u>and</u> $\varphi(0)\neq 0$. <u>Then</u> <u>there</u> <u>exists</u> <u>a</u> <u>unique</u> <u>function</u> $\psi \equiv \varphi^{-1}$, <u>which</u> <u>we</u> <u>call</u> <u>the</u>

<u>Stieltjes inverse of</u> φ, <u>such that</u> $\psi \in H^1$ <u>and</u>

$$\varphi * d\psi = h \quad \underline{\text{on}}\ (-\infty, \infty). \tag{1.10}$$

Finally, we state and prove a result on Stieltjes convolutions that is closely related to Theorem 1.5 in [1].

<u>Theorem</u> 1.3 (Sequences of Stieltjes convolutions). <u>Let</u> $\mathcal{R}$ <u>be bounded and let</u> $\{\varphi^n\}$ <u>be a sequence of functions such that</u>

(a) $\varphi^n \in H^{0,0}(\overline{\mathcal{R}})$ $(n = 1,2,\cdots\cdots)$;

(b) $\varphi^n \longrightarrow \varphi$ <u>as</u> $n \longrightarrow \infty$, <u>uniformly on</u> $\overline{\mathcal{R}}\times[0,T]$

<u>for every</u> $T \in [0,\infty)$. <u>Further, suppose</u> $\psi \in H^1$. <u>Then</u>, $\varphi^n * d\psi \longrightarrow \varphi * d\psi$ <u>as</u> $n \longrightarrow \infty$, <u>uniformly on</u> $\overline{\mathcal{R}}\times[0,T]$ <u>for every</u> $T \in [0,\infty)$.

<u>Proof</u>. From (a),(b) follows $\varphi \in C^0(\overline{\mathcal{R}}\times[0,\infty))$. Choose $T \in [0,\infty)$ and define a sequence of functions $\{\vartheta^n\}$ on $\overline{\mathcal{R}}\times(-\infty,\infty)$ by means of

$$\vartheta^n = |\theta^n * d\psi| \ , \quad \theta^n = \varphi^n - \varphi \quad (n = 1,2,\cdots). \tag{1.11}$$

By (a) and (1.11), $\theta^n \in H^{0,0}(\overline{\mathcal{R}})$. Moreover, it clearly suffices to show that, given $\delta > 0$, there exists an $N(\delta)$ such that

$$n > N(\delta) \Longrightarrow \vartheta^n(\underline{x},t) < \delta \ \text{for}\ (\underline{x},t) \in \overline{\mathcal{R}}\times[0,T]. \tag{1.12}$$

By (b) and (1.11), there exists $N(\delta)$ such that for fixed $T_0 > 0$,

$$n > N(\delta) \Longrightarrow |\theta^n(\underline{x},t)| < \delta/A \ \text{for}\ (\underline{x},t) \in \overline{\mathcal{R}}\times[0,T+T_0], \tag{1.13}$$

where $A = v_\psi(-T_0,T)$ is the total variation of ψ on $[-T_0,T]$. On the other hand, from (1.11) and a familiar estimate of Stieltjes

integrals ([10], p.232) follows

$$\vartheta^n(\underline{x},t) \leq \max_{\bar{\mathcal{R}}\times[0,T+T_0]} |\Theta^n| A \text{ for } (\underline{x},t)\in\bar{\mathcal{R}}\times[0,T]. \tag{1.14}$$

But (1.13) and (1.14) imply the desired conclusion (1.12).

We have so far considered only scalar-valued functions of position and time. Functions whose values are vectors or higher-order tensors will consistently be denoted by underlined letters. Thus, if the function $\underline{v}$ is defined on $\mathcal{R}\times\mathcal{J}$, its value $\underline{v}(\underline{x},t)$ at position $\underline{x}$ and time t is a known tensor for every $(\underline{x},t)\in\mathcal{R}\times\mathcal{J}$. Further, if the values of $\underline{v}$ are tensors of order $N \geq 1$, we write $v_{ij\ldots k}$ (N subscripts) for the components of $\underline{v}$ in a rectangular cartesian coordinate frame and henceforth adopt the usual summation convention for repeated subscripts. We say that $\underline{v}\in C^{M,N}(\mathcal{R}\times\mathcal{J})$ or that $\underline{v}\in H^{M,N}(\mathcal{R})$ if and only if the corresponding statements hold true for $v_{ij\ldots\ldots k}$. Suppose $\underline{u}$ and $\underline{v}$ are tensor-valued functions of the same order $N \geq 1$, while φ is scalar-valued, and let $\underline{u},\underline{v},\varphi$ have the same domain of definition. We then adopt the notation

$$\left.\begin{aligned} &\underline{u}*d\varphi = \underline{v} \iff u_{ij\ldots k}*d\varphi = v_{ij\ldots\ldots k}\,, \\ &\underline{u}*d\underline{v} = u_{ij\ldots\ldots k}*dv_{ij\ldots\ldots k}\,, \end{aligned}\right\} \tag{1.15}$$

provided the Stieltjes convolutions involved are meaningful.

We may now turn to preliminaries from the quasi-static linear theory of viscoelasticity. For this purpose let $\underline{u},\underline{\varepsilon}$, and $\underline{\sigma}$, with the components u_i,ε_{ij}, and σ_{ij}, be the field histories

of displacement, infinitesimal strain, and stress, respectively. All of these field histories are to be regarded as functions of position and time, defined on $\bar{R}\times(-\infty,\infty)$, where R is the (regular) region of space occupied by the interior of the body in its undeformed state. We assume the body to be originally undisturbed in the sense of the <u>initial conditions</u>

$$u_i = \varepsilon_{ij} = \sigma_{ij} = 0 \quad \text{on} \quad \bar{R}\times(-\infty,0). \tag{1.16}$$

Next, we recall the relevant <u>fundamental system of field equations</u>, which must hold throughout the space-time domain $R\times(-\infty,\infty)$. The linearized <u>displacement-strain relations</u> take the form

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} , \tag{1.17}$$

while the <u>stress equations of equilibrium</u> become

$$\sigma_{ij,j} + F_i = 0 , \quad \sigma_{ji} = \sigma_{ij} , \tag{1.18}$$

where F_i stands for the components of the body-force density $\underline{F}$. In stating the <u>stress-strain relations</u> we shall make use of the deviatoric components of stress and strain defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\,\delta_{ij}\sigma_{kk} , \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\,\delta_{ij}\varepsilon_{kk} , \tag{1.19}$$

in which δ_{ij} denotes the Kronecker delta. Furthermore, we shall confine our attention to the isothermal relaxation integral law appropriate to linear, homogeneous and isotropic, viscoelastic solids. This constitutive law may now be written as

$$s_{ij} = e_{ij}*dG_1 , \quad \sigma_{kk} = \varepsilon_{kk}*dG_2 , \tag{1.20}$$

provided G_1 and G_2, which are functions of the time alone, designate the respective relaxation moduli in shear and isotropic compression.

To the foregoing initial conditions and field equations we adjoin the boundary conditions. From here on let

$$S_i = \sigma_{ij} n_j \tag{1.21}$$

denote the components of the traction vector $\underline{S}$ on a surface with the outward unit normal vector $\underline{n}$. The <u>boundary conditions</u> governing the standard <u>mixed problem</u> then appear as

$$u_i = \hat{u}_i \text{ on } B_1 \times (-\infty, \infty) \ , \ S_i = \hat{S}_i \text{ on } B_2 \times (-\infty, \infty), \tag{1.22}$$

in which B_1 and B_2 are complementary subsets of the boundary B of R, while $\hat{\underline{u}}$ and $\hat{\underline{S}}$ represent prescribed surface displacements and surface tractions. In the <u>first boundary-value problem</u> B_2 is empty and the displacements are given on $B \times (-\infty, \infty)$; in the <u>second boundary-value problem</u> B_1 is empty, the surface tractions being assigned throughout $B \times (-\infty, \infty)$.

With a view toward an economical statement of the hypotheses underlying subsequent theorems we introduce

<u>Definition</u> 1.3 (States, regular viscoelastic states). <u>If u is a vector-valued — and ε, σ are second-order tensor-valued functions of position and time defined on $\mathcal{R} \times (-\infty, \infty)$, we call the ordered array $\mathcal{J} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}]$ a state on $\mathcal{R} \times (-\infty, \infty)$ and denote by</u> $\mathcal{J}(\underline{x}, t) = [\underline{u}(\underline{x}, t), \underline{\varepsilon}(\underline{x}, t), \underline{\sigma}(\underline{x}, t)]$ <u>the value of</u> $\mathcal{J}$ <u>at</u> $(\underline{x}, t)$.

<u>We say that</u> $\mathcal{J} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}]$ <u>is a regular viscoelastic state on</u> $\mathcal{R} \times (-\infty, \infty)$ <u>corresponding to the relaxation functions</u> $G_\nu (\nu = 1, 2)$

and the body-force density $\underline{F}$, *and write*

$$\mathcal{S} = [\underline{u},\underline{\varepsilon},\underline{\sigma}] \in \mathcal{V}(\mathcal{R},G_1,G_2,\underline{F}), \tag{1.23}$$

provided:

(a) $\underline{u} \in H^{2,1}(\mathcal{R})$ *and* $\underline{\varepsilon},\underline{\sigma},\underline{F} \in H^{0,0}(\mathcal{R})$, *while* $G_\nu \in H^2$ *with* $G_\nu(0) > 0$ $(\nu = 1,2)$;

(b) $\underline{u},\underline{\varepsilon},\underline{\sigma},\underline{F},G_\nu(\nu = 1,2)$ *on the interior*[4] *of* $\mathcal{R}\times(-\infty,\infty)$ *satisfy the field equations* (1.17), (1.18), (1.19), (1.20);

(c) *if* $\mathcal{R}$ *is an exterior region, then, as* $x \equiv |\underline{x}| \rightarrow \infty$,

$$\left.\begin{aligned} &\underline{u}(\underline{x},\cdot) = O(x^{-1}) \ , \ \dot{\underline{u}}(\underline{x},\cdot) = O(x^{-1}), \\ &\underline{\sigma}(\underline{x},\cdot) = O(x^{-2}) \ , \ \underline{F}(\underline{x},\cdot) = O(x^{-3}), \end{aligned}\right\} \tag{1.24}$$

uniformly on $[0,T]$ *for every* $T \in [0,\infty)$.

If, in particular, $\underline{F} = 0$ *on* $\mathcal{R}\times(-\infty,\infty)$, *we write*

$$\mathcal{S} = [\underline{u},\underline{\varepsilon},\underline{\sigma}] \in \mathcal{V}(\mathcal{R},G_1,G_2). \tag{1.25}$$

In (1.24), as in the sequel, the notion of "order of magnitude" is used in its standard mathematical connotation. Thus, if $\underline{v}$ is defined on $\mathcal{R}\times[0,\infty)$, $\mathcal{R}$ being an exterior region, we write $\underline{v}(\underline{x},\cdot) = O(x^n)$ as $x \rightarrow \infty$, uniformly on $[0,T]$ for every $T \in [0,\infty)$ if and only if : given $T \geq 0$, there exist numbers $\rho(T)$ and $M(T)$ such that $x \equiv |\underline{x}| > \rho$ implies $|v_{ij.....k}(\underline{x},t)| < Mx^n$ for every $t \in [0,T]$.

Conditions (a) and (b), which are partly redundant but mutually consistent, could be weakened somewhat at the expense

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[4] Recall that $\mathcal{R}$ may be open or closed.

of more elaborate smoothness assumptions. Note that the field histories belonging to a regular viscoelastic state may possess finite jump-discontinuities with respect to time at t=0.

Addition of states and multiplication of a state by a scalar constant are defined as follows. Suppose $\mathcal{J} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]$ and $\mathcal{J}' = [\underline{u}',\underline{\varepsilon}',\underline{\sigma}']$ are states on $\bar{R}\times(-\infty,\infty)$. Then

$$\left.\begin{aligned} \mathcal{J} + \mathcal{J}' &= [\underline{u}+\underline{u}',\underline{\varepsilon}+\underline{\varepsilon}',\underline{\sigma}+\underline{\sigma}'], \\ \lambda\mathcal{J} &= [\lambda\underline{u},\lambda\underline{\varepsilon},\lambda\underline{\sigma}]. \end{aligned}\right\} \quad (1.26)$$

Further, we define the derivative of a state $\mathcal{J} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]$ with respect to the k-th cartesian coordinate (in a given coordinate frame) by means of

$$\mathcal{J}_{,k} = [\underline{u}_{,k},\underline{\varepsilon}_{,k},\underline{\sigma}_{,k}], \quad (1.27)$$

provided the requisite space-derivatives $u_{i,k}$, $\varepsilon_{ij,k}$, and $\sigma_{ij,k}$ exist.

The uniqueness and the reciprocal theorem of linear viscoelasticity play a particularly essential part in the analysis to follow. For this reason we include here statements of both of these theorems.

<u>Theorem</u> 1.4 (Uniqueness theorem). <u>Suppose</u>

$$\left.\begin{aligned} \mathcal{J} &= [\underline{u},\underline{\varepsilon},\underline{\sigma}] \in \mathcal{V}(\bar{R},G_1,G_2,\underline{F}), \\ \mathcal{J}' &= [\underline{u}',\underline{\varepsilon}',\underline{\sigma}'] \in \mathcal{V}(\bar{R},G_1,G_2,\underline{F}). \end{aligned}\right\} \quad (1.28)$$

<u>Further, let</u>

$$\underline{u} = \underline{u}' \text{ on } B_1\times(-\infty,\infty), \quad \underline{S} = \underline{S}' \text{ on } B_2\times(-\infty,\infty), \quad (1.29)$$

where B_1 and B_2 are complementary subsets of B. Then

$$[\underline{u},\underline{\varepsilon},\underline{\sigma}] = [\underline{u}',\underline{\varepsilon}',\underline{\sigma}'] + [\underline{w},0,0] \text{ on } \bar{R}\times(-\infty,\infty), \quad (1.30)$$

where, for every $(\underline{x},t)\in\bar{R}\times(-\infty,\infty)$,

$$\underline{w}(\underline{x},t) = \underline{a}(t) + \underline{\omega}(t)\wedge\underline{x} \quad \text{with} \quad \underline{a},\ \underline{\omega}\in H^1, \quad (1.31)^5$$

so that w represents an (infinitesimal) rigid motion of the entire body.

Volterra's [11] proof of the preceding uniqueness theorem was spelled out in detail in [1] (Theorem 8.1) with limitation to bounded regular regions of space. The extension of this proof to unbounded (exterior) regions offers no difficulties in the presence of the regularity assumptions (1.24) which imply further that $\underline{\varepsilon}(\underline{x},\cdot) = O(x^{-2})$ as $x\to\infty$, uniformly on every interval $[0,T]$.

Theorem 1.5 (Reciprocal theorem). Let

$$\left.\begin{aligned} \mathcal{S} &= [\underline{u},\underline{\varepsilon},\underline{\sigma}]\in\mathcal{V}(\bar{R},G_1,G_2,\underline{F}),\\ \mathcal{S}' &= [\underline{u}',\underline{\varepsilon}',\underline{\sigma}']\in\mathcal{V}(\bar{R},G_1,G_2,\underline{F}'). \end{aligned}\right\} \quad (1.32)$$

Then, for each $t\in(-\infty,\infty)$,

$$\begin{aligned} &\int_B [\underline{S}*d\underline{u}'](\underline{x},t)dA + \int_R [\underline{F}*d\underline{u}'](\underline{x},t)dV = \\ &\int_B [\underline{S}'*d\underline{u}](\underline{x},t)dA + \int_R [\underline{F}'*d\underline{u}](\underline{x},t)dV = \\ &\int_R [\underline{\sigma}*d\underline{\varepsilon}'](\underline{x},t)dV = \int_R [\underline{\sigma}'*d\underline{\varepsilon}](\underline{x},t)dV\ . \end{aligned} \quad (1.33)$$

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5 Throughout this paper the symbols "." and "$\wedge$" are used to indicate scalar and vectorial multiplication of vectors, respectively.

A proof of this theorem, valid for bounded regions, appears in [1] (Theorem 7.4); its generalization to exterior regions is a trivial matter.

Finally, we cite a result that furnishes an extension to viscoelasticity theory of the Papkovich-Neuber stress functions in classical elastostatics (see Theorem 9.2 of [1]).

Theorem 1.6 (Generalized Papkovich-Neuber solution).

(a) *Let* $\mathcal{R}$ *be open and bounded. Let* $\underline{F} \in H^{1,1}(\mathcal{R})$ *be vector-valued and assume* $G_\nu \in H^2$ *with* $G_\nu(0) > 0$ $(\nu=1,2)$;

(b) *Suppose* $\varphi \in H^{3,1}(\mathcal{R})$ *and* $\underline{\psi} \in H^{3,1}(\mathcal{R})$ *are a real-valued and a vector-valued function, respectively, such that*

$$\nabla^2\varphi = -\frac{1}{2}\,\underline{x}\cdot\underline{f}\ , \quad \nabla^2\underline{\psi} = \frac{1}{2}\,\underline{f} \quad \text{on } \mathcal{R}\times(-\infty,\infty), \tag{1.34}$$ [6]

where

$$\underline{f} = \underline{F}*dG_1^{-1}*d(2G_1+G_2)^{-1}; \tag{1.35}$$

(c) *Define a state* $[\underline{u},\underline{\varepsilon},\underline{\sigma}]$ *on* $\mathcal{R}\times(-\infty,\infty)$ *by means of*

$$\underline{u} = \nabla(\varphi+\underline{x}\cdot\underline{\psi})*d(G_1+2G_2) - 4\underline{\psi}*d(2G_1+G_2) \tag{1.36}$$

in conjunction with (1.17) *and* (1.19),(1.20).

Then

$$\mathcal{J} = [\underline{u},\underline{\varepsilon},\underline{\sigma}] \in \mathcal{V}(\mathcal{R},G_1,G_2,\underline{F}). \tag{1.37}$$

The completeness of the foregoing solution to the field equations was established in Theorem 9.4 of [1].

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[6] ∇ is the spatial gradient operator and ∇^2 the spatial Laplacian operator.

2. Kelvin's problem in viscoelasticity theory: the basic singular state. Higher-order singular states.

In the current section we deal first with the problem of a concentrated load applied at a point of a viscoelastic medium that occupies the entire space E. This problem is the counterpart in viscoelasticity theory of Kelvin's problem in elastostatics and its solution supplies the fundamental singular solution of the field equations under present consideration.

Kelvin's solution of his problem, which was first published without proof in [12] (1884), was later deduced by Kelvin and Tait ([13], p.227 et seq.) through a limit process that takes as its point of departure an elastic solid subjected to distributed body forces. This limit definition of the notion of an internal concentrated load in classical elastostatics is sketched in somewhat greater detail by Love [5] (Art.130) and was made fully explicit in [8], which may serve as a guide for the generalization to be attempted here. The theorem to which we turn now is intended to serve a dual purpose: first, it aims at a mathematically precise and physically natural unique characterization through a limit process of the singular problem at hand; second, it furnishes the explicit solution to the problem thus formulated. To facilitate the statement of this theorem we first agree that $\Omega_\rho(\underline{x}^o)$ denotes an open sphere with the radius ρ centered at $\underline{x}^o$ and adopt

Definition 2.1 (Limit definition of a concentrated load). *We say that $\{\underline{F}^n\}$ is a sequence of body-force distributions that tends to*

a concentrated load $\underline{L}$ applied at $\underline{x}^{o}$ if $\underline{x}^{o} \in E$ and $\underline{L} \in H^{1}$ is vector-valued, while $\{\underline{F}^{n}\}$ has the following properties:

(a) for every n $(n=1,2,\cdots)$ $\underline{F}^{n}$ is a vector-valued function with

$$\underline{F}^{n} \in H^{2,1}(E)\ ,\ \underline{F}^{n} = 0 \ \text{ on } \ (E-\Omega^{n})\times(-\infty,\infty), \qquad (2.1)$$

where $\{\Omega^{n}\}$ is a sequence of "load regions" characterized by

$$\Omega^{n} = \Omega_{\delta^{n}}(\underline{x}^{o})\ ,\ \delta^{n} \to 0 \ \text{ as } \ n \to \infty; \qquad (2.2)$$

(b) $\int_{\Omega^{n}} \underline{F}^{n}(\underline{x},\cdot)dV \to \underline{L}$ as $n \to \infty$, uniformly on $[0,T]$ for every $T \in [0,\infty)$;

(c) the sequence of functions $\{\Phi^{n}\}$, defined by

$$\Phi^{n} = \int_{\Omega^{n}} |\underline{F}^{n}(\underline{x},\cdot)|\, dV \ \text{ on } \ (-\infty,\infty) \quad (n=1,2,\cdots), \qquad (2.3)$$

is uniformly bounded on $[0,T]$ for every $T \in [0,\infty)$.

We are now in a position to turn to

Theorem 2.1 (Viscoelastic Kelvin-state). Let $\{\underline{F}^{n}\}$ be a sequence of body-force distributions that tends to a concentrated load $\underline{L}$ applied at $\underline{x}^{o}$. Let $G_{\nu} \in H^{2}$ with $G_{\nu}(0) > 0$ $(\nu=1,2)$. Then:

(a) there exists a unique sequence of states $\{\mathcal{J}^{n}\}$ such that

$$\mathcal{J}^{n} = [\underline{u}^{n},\underline{\epsilon}^{n},\underline{\sigma}^{n}] \in \mathcal{V}(E,G_{1},G_{2},\underline{F}^{n}) \quad (n=1,2,\cdots); \qquad (2.4)$$

(b) $\mathcal{J}^{n}$ converges to a limit state $\mathcal{J}$ as $n \to \infty$, uniformly on $\bar{\mathcal{R}}\times(-\infty,T]$ for every bounded $\bar{\mathcal{R}}$ such that $\underline{x}^{o} \notin \bar{\mathcal{R}}$ and every $T \in (-\infty,\infty)$;

(c) *the limit-state* $\mathscr{S} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}]$ *is independent of the particular choice of the sequence* $\{\underline{F}^n\}$ *and — in the sense of Theorem* 1.6 — *is generated by the stress functions* φ, $\underline{\psi}$ *defined through*

$$\varphi(\underline{x},t) = 0 \;,\; \underline{\psi}(\underline{x},t) = -\frac{\underline{f}(t)}{8\pi|\underline{x}-\underline{x}^{o}|} \tag{2.5}$$

for every $\underline{x} \neq \underline{x}^{o}$ *and every* $t \in (-\infty,\infty)$, *where*

$$\underline{f} = \underline{L} * dG_1^{-1} * d(2G_1+G_2)^{-1} \text{ on } (-\infty,\infty). \tag{2.6}$$

We call $\mathscr{S}$ *the (viscoelastic) Kelvin-state corresponding to a concentrated load* $\underline{L}$ *applied at* $\underline{x}^{o}$ *and to the relaxation functions* G_1, G_2.

Proof. Note that the existence of the Stieltjes inverses G_1^{-1}, $(2G_1+G_2)^{-1}$ is assured by Theorem 1.2 and the present hypotheses on $G_\nu (\nu=1,2)$. Define two sequences of functions $\{\varphi^n\}$ and $\{\underline{\psi}^n\}$ by setting, for every $n(n=1,2,\cdots)$ and all $(\underline{x},t) \in E\times(-\infty,\infty)$,

$$\left.\begin{aligned} \varphi^n(\underline{x},t) &= \frac{1}{8\pi}\int_{\Omega^n} \frac{\underline{\xi}\cdot\underline{f}^n(\underline{\xi},t)}{|\underline{x}-\underline{\xi}|}\, dV_{\underline{\xi}} \;, \\ \underline{\psi}^n(\underline{x},t) &= \frac{-1}{8\pi}\int_{\Omega^n} \frac{\underline{f}^n(\underline{\xi},t)}{|\underline{x}-\underline{\xi}|}\, dV_{\underline{\xi}} \;, \end{aligned}\right\} \tag{2.7}$$ [7]

where

$$\underline{f}^n = \underline{F}^n * dG_1^{-1} * d(2G_1+G_2)^{-1} \quad \text{on} \quad E\times(-\infty,\infty). \tag{2.8}$$

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[7] A subscript attached to an "element of volume" or an "element of area" in a volume or surface integral indicates the appropriate space variable of integration.

Thus $\varphi^n(\cdot,t)$ and $\underline{\psi}^n(\cdot,t)$ are Newtonian potentials of mass distributions over $\overline{\Omega}^n$. It follows from (a) in Definition 2.1, Theorem 1.1, and a trivial extension of Lemma 9.1 in [1] that φ^n, $\underline{\psi}^n \in H^{3,1}(E)$ and that

$$\nabla^2\varphi^n = -\frac{1}{2}\,\underline{x}\cdot\underline{f}^n\ ,\quad \nabla^2\underline{\psi}^n = \frac{1}{2}\,\underline{f}^n \text{ on } E\times(-\infty,\infty). \tag{2.9}$$

Consequently and because of the behavior of the integrals in (2.7) as $x \longrightarrow \infty$, the functions φ^n and $\underline{\psi}^n$, when used as stress functions in conjunction with Theorem 1.6, generate a state $\mathcal{S}^n = [\underline{u}^n,\underline{\varepsilon}^n,\underline{\sigma}^n]$ that meets (2.4). The uniqueness of $\mathcal{S}^n$ is immediate from Theorem 1.4. Thus part (a) of the theorem under consideration has been confirmed.

Let $\mathcal{S} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]$ be the state generated by the stress functions $\varphi,\underline{\psi}$ in (2.5). Choose $\overline{\mathcal{R}}$ such that $\underline{x}^o \notin \overline{\mathcal{R}}$, choose $T \in [0,\infty)$, and hold $\overline{\mathcal{R}}\times[0,T]$ fixed for the remainder of the argument. In order to establish parts (b) and (c) of the theorem it suffices[8] to show that, as $n \longrightarrow \infty$,

$$\underline{u}^n \longrightarrow \underline{u}\ ,\ \underline{\varepsilon}^n \longrightarrow \underline{\varepsilon}\ ,\ \underline{\sigma}^n \longrightarrow \underline{\sigma} \text{ uniformly on } \overline{\mathcal{R}}\times[0,T]. \tag{2.10}$$

By virtue of Theorems 1.6, 1.3, however, (2.10) follows if we show that for a fixed choice of the coordinate frame, as $n \longrightarrow \infty$ and uniformly on $\overline{\mathcal{R}}\times[0,T]$,

[8] Observe that both $\mathcal{S}$ and $\mathcal{S}^n$ vanish identically on $\overline{\mathcal{R}}\times(-\infty,0)$.

$$\left.\begin{array}{l}\varphi^n \to \varphi\ ,\ \varphi^n_{,i} \to \varphi_{,i}\ ,\ \varphi^n_{,ij} \to \varphi_{,ij}\ , \\ \underline{\psi}^n \to \underline{\psi}\ ,\ \underline{\psi}^n_{,i} \to \underline{\psi}_{,i}\ ,\ \underline{\psi}^n_{,ij} \to \underline{\psi}_{,ij}\ .\end{array}\right\} \quad (2.11)$$

Since all of (2.11) may be established by strictly analogous means, we shall demonstrate merely that

$$\underline{\psi}^n \to \underline{\psi} \text{ as } n \to \infty, \text{ uniformly on } \bar{\mathcal{R}}\times[0,T]. \quad (2.12)$$

To this end we infer from (2.5),(2.7) that for $(\underline{x},t)\in\bar{\mathcal{R}}\times[0,T]$,

$$8\pi[\underline{\psi}^n(\underline{x},t) - \underline{\psi}(\underline{x},t)] = \underline{I}^n_1(\underline{x},t) + \underline{I}^n_2(\underline{x},t), \quad (2.13)$$

provided

$$\left.\begin{array}{l}\underline{I}^n_1(\underline{x},t) = \displaystyle\int_{\Omega^n} \left[\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}\right]\underline{f}^n(\underline{\xi},t)dV_{\underline{\xi}}\ , \\ \underline{I}^n_2(\underline{x},t) = \dfrac{1}{|\underline{x}-\underline{x}^o|}\left[\underline{f}(t) - \displaystyle\int_{\Omega^n} \underline{f}^n(\underline{\xi},t)dV\right]\ .\end{array}\right\} \quad (2.14)$$

Accordingly, it is sufficient to show that

$$\underline{I}^n_1,\underline{I}^n_2 \to 0 \quad \text{as} \quad n \to \infty, \text{ uniformly on } \bar{\mathcal{R}}\times[0,T]. \quad (2.15)$$

In view of (2.8), the first of (2.14) becomes

$$\underline{I}^n_1(\underline{x},t) = \int_{\Omega^n} \left[\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}\right][\underline{F}^n * dG](\underline{\xi},t)dV_{\underline{\xi}} \quad (2.16)$$

for all $(\underline{x},t)\in\bar{\mathcal{R}}\times[0,T]$, if one sets

$$G = G_1^{-1} * d(2G_1+G_2)^{-1} \text{ on } (-\infty,\infty). \quad (2.17)$$

Now, by the hypothesis underlying the present theorem and (a) in Definition 2.1, there is an $N > 0$ such that $\bar{\mathcal{R}}\cap\Omega^n$ is empty when $n > N$. Consequently, for $n > N$ and $(\underline{x},t)\in\bar{\mathcal{R}}\times[0,T]$, the volume

integral (2.16) is proper. This entitles us to reverse the order of the process of space-integration and convolution in (2.16) when $n > N$, as is readily seen with the aid of Theorem 1.2, (a) and (g) in Theorem 1.1, and the available regularity of the functions $\underline{F}^n$ and $G_\nu(\nu=1,2)$. Hence

$$\underline{I}_1^n = \underline{I}^n * dG \quad \text{on} \quad \bar{\mathcal{R}}\times[0,T] \quad (n > N), \tag{2.18}$$

where

$$\underline{I}^n(\underline{x},t) = \int_{\Omega^n} [\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}]\underline{F}^n(\underline{\xi},t)dV_{\underline{\xi}} \tag{2.19}$$

and clearly $\underline{I}^n$ is continuous on $\bar{\mathcal{R}}\times[0,T]$. From (2.18) follows the estimate, valid for $n > N$ and fixed $T_o > 0$,

$$|\underline{I}_1^n| \le \max_{\bar{\mathcal{R}}\times[0,T+T_o]} |\underline{I}^n| \, \mathcal{V}_G(-T_o,T) \quad \text{on } \bar{\mathcal{R}}\times[0,T], \tag{2.20}$$

where $\mathcal{V}_G(-T_o,T)$ is the total variation of G on $[-T_o,T]$. On the other hand, one draws from (2.19) that

$$|\underline{I}^n(\underline{x},t)| \le \max_{\underline{\xi}\in\bar{\Omega}^n} \left|\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}\right| \int_{\Omega^n} |\underline{F}^n(\underline{\xi},t)|\,dV \tag{2.21}$$

for every $(\underline{x},t)\in\bar{\mathcal{R}}\times[0,T+T_o]$. But, in view of (c) in Definition 2.1, the integral in (2.21) is bounded uniformly for all n and all $t\in[0,T+T_o]$, whereas the coefficient of this integral tends to zero uniformly as $n \to \infty$ for all $\underline{x}\in\bar{\mathcal{R}}$ because of (a) in Definition 2.1. Consequently $\underline{I}^n \to 0$, as $n \to \infty$, uniformly on $\bar{\mathcal{R}}\times[0,T+T_o]$ and hence (2.20) implies the statement concerning $\underline{I}_1^n$ in (2.15).

integral (2.16) is proper. This entitles us to reverse the order of the process of space-integration and convolution in (2.16) when $n > N$, as is readily seen with the aid of Theorem 1.2, (a) and (g) in Theorem 1.1, and the available regularity of the functions $\underline{F}^n$ and $G_\nu (\nu=1,2)$. Hence

$$\underline{I}_1^n = \underline{I}^n * dG \quad \text{on} \quad \bar{R}\times[0,T] \quad (n > N), \tag{2.18}$$

where

$$\underline{I}^n(\underline{x},t) = \int_{\Omega^n} \left[\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}\right]\underline{F}^n(\underline{\xi},t)dV_{\underline{\xi}} \tag{2.19}$$

and clearly $\underline{I}^n$ is continuous on $\bar{R}\times[0,T]$. From (2.18) follows the estimate, valid for $n > N$ and fixed $T_o > 0$,

$$|\underline{I}_1^n| \leq \max_{\bar{R}\times[0,T+T_o]} |\underline{I}^n| \, \mathcal{V}_G(-T_o,T) \text{ on } \bar{R}\times[0,T], \tag{2.20}$$

where $\mathcal{V}_G(-T_o,T)$ is the total variation of G on $[-T_o,T]$. On the other hand, one draws from (2.19) that

$$|\underline{I}^n(\underline{x},t)| \leq \max_{\underline{\xi}\in\bar{\Omega}^n}\left|\frac{1}{|\underline{x}-\underline{x}^o|} - \frac{1}{|\underline{x}-\underline{\xi}|}\right| \int_{\Omega^n}|\underline{F}^n(\underline{\xi},t)|\,dV \tag{2.21}$$

for every $(\underline{x},t)\in\bar{R}\times[0,T+T_o]$. But, in view of (c) in Definition 2.1, the integral in (2.21) is bounded uniformly for all n and all $t\in[0,T+T_o]$, whereas the coefficient of this integral tends to zero uniformly as $n \to \infty$ for all $\underline{x}\in\bar{R}$ because of (a) in Definition 2.1. Consequently $\underline{I}^n \to 0$, as $n \to \infty$, uniformly on $\bar{R}\times[0,T+T_o]$ and hence (2.20) implies the statement concerning $\underline{I}_1^n$ in (2.15).

By virtue of (2.6),(2.8), and (2.17), the second of (2.14), after a permissible reversal of the processes of space-integration and convolution, yields

$$\underline{I}_2^n(\underline{x},t) = \frac{1}{|\underline{x}-\underline{x}^o|}[\{\underline{L} - \int_{\Omega^n} \underline{F}^n(\underline{\xi},\cdot)dV\}*dG](\underline{x},t) \tag{2.22}$$

for all $(\underline{x},t)\in\bar{\mathcal{R}}\times[0,T]$. The assertion concerning $\underline{I}_2^n$ in (2.15) now follows from (b) in Definition 2.1 together with Theorem 1.3. This completes the proof.

Conditions (a) and (c) in Definition 2.1 trivially imply

(c') $\int_{\Omega^n} (\underline{x}-\underline{x}^o)\wedge\underline{F}^n(\underline{x},\cdot)dV \longrightarrow 0$ as $n\longrightarrow\infty$, uniformly on $[0,T]$

for every $T\in[0,\infty)$. It is natural to ask whether the seemingly artificial requirement (c) may be replaced by (c') without impairing the conclusion in Theorem 2.1. That this is not possible is clear from Theorem 4.3 in [8], on passing to the special case of the elastic solid. Indeed, if (c) is replaced by (c') in Definition 2.1, one can construct a sequence $\{\underline{F}^n\}$ of body-force distributions that tends to a load $\underline{L} = 0$ at $\underline{x}^o$ and yet generates — in the sense of the limit process underlying Theorem 1.2 — a limit state other than the null-state. Requirement (c) does become superfluous, however, if $\{\underline{F}^n\}$ is restricted to body forces that are, at every instant, parallel and of the same sense. In this special case one has

$$\underline{F}^n(\underline{x},t) = |\underline{F}^n(\underline{x},t)|\,\underline{k}(t) \text{ for all } (\underline{x},t)\in\Omega^n\times[0,\infty), \tag{2.23}$$

$\underline{k}(t)$ being a unit-vector, and (a),(b) are readily found to imply (c), in Definition 2.1.

From here on let $\underline{e}_\alpha$ denote a unit-vector in the x_α-direction. If $\underline{L} = h\underline{e}_\alpha$ in Theorem 2.1, we call the limit state $\mathcal{S}$ the <u>normalized</u> (viscoelastic) <u>Kelvin-state</u> (corresponding to a load in the x_α-direction applied at $\underline{x}^o$ and to the relaxation functions G_1, G_2). Moreover, we denote the value of this state at $(\underline{x}, t)$ by

$$\mathcal{S}^\alpha(\underline{x},t;\underline{x}^o) = [\underline{u}^\alpha(\underline{x},t;\underline{x}^o), \underline{\varepsilon}^\alpha(\underline{x},t;\underline{x}^o), \underline{\sigma}^\alpha(\underline{x},t;\underline{x}^o)] \tag{2.24}$$

with the understanding that all superscripts not otherwise specified henceforth have the range of the integers (1,2,3). Evidently, for every constant vector $\underline{a}$,

$$\mathcal{S}^\alpha(\underline{x},t;\underline{x}^o+\underline{a}) = \mathcal{S}^\alpha(\underline{x}-\underline{a},t;\underline{x}^o), \tag{2.25}$$

whence in particular

$$\mathcal{S}^\alpha(\underline{x},t;\underline{x}^o) = \mathcal{S}^\alpha(\underline{x}-\underline{x}^o,t;0). \tag{2.26}$$

We now record the cartesian components of displacement and stress belonging to $\mathcal{S}^\alpha(\underline{x},t;0)$. These components are easily obtained on setting $\underline{L} = h\underline{e}_\alpha$ and $\underline{x}^o = 0$ in (2.5), by recourse to (1.36), (1.17), (1.19), and (1.20). In this manner one arrives at

$$\left.\begin{aligned} u_i^\alpha(\underline{x},t;0) &= \frac{1}{8\pi x}\{2J_1(t)\,[\delta_{\alpha i} + \frac{x_\alpha x_i}{x^2}] + 3Q_1(t)[\delta_{\alpha i} - \frac{x_\alpha x_i}{x^2}]\}, \\ \sigma_{ij}^\alpha(\underline{x},t;0) &= \frac{-3}{8\pi x^3}\{\frac{x_\alpha x_i x_j}{x^2}[2h(t) - 3Q_2(t)] \\ &\qquad + Q_2(t)[\delta_{\alpha i}x_j + \delta_{\alpha j}x_i - \delta_{ij}x_\alpha]\}, \end{aligned}\right\} \tag{2.27}$$

where $x = |\underline{x}| = \sqrt{x_k x_k}$, as before, and J_1, Q_1, Q_2 are auxiliary response functions defined by

$$J_1 = G_1^{-1}\ ,\ Q_1 = (2G_1+G_2)^{-1}\ ,\ Q_2 = Q_1 * dG_1 \text{ on } (-\infty,\infty). \quad (2.28)^9$$

In the particular case of an elastic solid one has

$$\left.\begin{aligned} &G_1 = 2\mu h\ ,\ G_2 = 3\varkappa h\ ,\\ &J_1 = \frac{h}{2\mu}\ ,\ Q_1 = \frac{h}{4\mu+3\varkappa}\ ,\ Q_2 = \frac{2\mu h}{4\mu+3\varkappa}\ , \end{aligned}\right\} \quad (2.29)$$

provided μ and $\varkappa$ stand for the shear modulus and the bulk modulus of the material, respectively. In this instance (2.27), for all $\underline{x} \neq 0$ and $0 \leq t < \infty$, reduce to the analogous formulas appropriate to the normalized <u>elastostatic</u> Kelvin-state (see, for example, equations (5.4) in [8]).

We take up next a theorem that summarizes certain relevant properties of $\mathcal{J}^{\alpha}(\underline{x},t;0)$. For the sake of convenience in stating this and subsequent results we introduce the symbol $E'_{\underline{x}^o}$ for the open region consisting of all points of the Euclidean space E with the exception of the point $\underline{x}^o$. Further, we shall simply write E' in place of $E'_{\underline{x}^o}$ when $\underline{x}^o = 0$, so that E' denotes the complement of the origin with respect to the entire space E.

<u>Theorem</u> 2.2 (Properties of the Kelvin-state). <u>The normalized Kelvin-state</u> $\mathcal{J}^{\alpha}(\underline{x},t;0)$ <u>has the properties</u>:

(a) $\underline{u}^{\alpha},\underline{\varepsilon}^{\alpha},\underline{\sigma}^{\alpha} \in H^{\infty,1}(E')$ <u>and</u> $\mathcal{J}^{\alpha} = [\underline{u}^{\alpha},\underline{\varepsilon}^{\alpha},\underline{\sigma}^{\alpha}] \in \mathcal{V}(E',G_1,G_2]$;

(b) $\int_{\Pi} \underline{S}^{\alpha}(\underline{x},\cdot;0)dA = \underline{e}_{\alpha}h$ <u>on</u> $(-\infty,\infty)$,

<u>where</u> Π <u>is any closed regular surface surrounding the origin and</u>

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9 Note that J_1 is the creep compliance in shear (see Theorem 3.3 in [1]).

$\underline{S}^{\alpha}$ *is the traction vector of* $\mathcal{S}^{\alpha}$ *on that side of* Π *which faces the origin;*

(c) $\int_{\Pi} \underline{x} \wedge \underline{S}^{\alpha}(\underline{x},\cdot;0)dA = 0$ *on* $(-\infty,\infty)$;

(d) $\underline{u}^{\alpha}(\underline{x},\cdot;0) = O(x^{-1})$, $\underline{\sigma}^{\alpha}(\underline{x},\cdot;0) = O(x^{-2})$ *as* $x \rightarrow 0$, *uniformly on* $[0,T]$ *for every* $T \in [0,\infty)$.

Proof. Property (a) is immediate from the specific form of the stress functions generating $\mathcal{S}^{\alpha}$, exhibited in (2.5), and from Theorem 1.6. To confirm (b), use the second of (2.27) and note that because of (a) the surface Π may be restricted to be a sphere centered at the origin. Property (d), which characterizes the order of the displacement and stress singularities at the load-point $\underline{x}^{o} = 0$, follows at once from (2.27), whereas (c) is readily found to be implied by (a) and (d).

In the treatise literature on elasticity theory the original Kelvin-problem is frequently formulated on the basis of the elastostatic counterpart of (a),(b),(c) in Theorem 2.2. As was pointed out in [8], such a direct formulation of the problem fails to determine its solution uniquely. Similarly, (a),(b),(c) do not furnish a unique characterization of the viscoelastic Kelvin-state $\mathcal{S}^{\alpha}(\underline{x},t;0)$ since — as we shall have occasion to see shortly — there exist viscoelastic states (distinct from the null-state) that are regular on $E' \times (-\infty,\infty)$ and possess self-equilibrated singularities at the origin. In this connection we mention further that the viscoelastic Kelvin-state $\mathcal{S}^{\alpha}(\underline{x},t;0)$, whose displacements and stresses are exhibited

in (2.27), may alternatively be obtained directly from the corresponding elastostatic state through a purely formal application of the well-known correspondence principle that links the linear theories of elasticity and viscoelasticity.[10] However, this elementary method for deducing the viscoelastic Kelvin-state does not assure the truth of Theorem 2.1, from which the singular state in question derives its intrinsic physical significance.

We turn now to a discussion of the higher-order singular viscoelastic states that may be generated through a single space-differentiation of the Kelvin-state. Thus let $\mathcal{S}^{\alpha}(\underline{x},t;\underline{x}^{o})$ once again be the normalized viscoelastic Kelvin-state introduced previously and define a set of nine <u>viscoelastic</u> <u>doublet</u>-<u>states</u>

$$\mathcal{S}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}) = [\underline{u}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}),\underline{\varepsilon}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}),\underline{\sigma}^{\alpha\beta}(\underline{x},t;\underline{x}^{o})] \quad (2.30)$$

by means of

$$\mathcal{S}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}) = \frac{\partial}{\partial x_{\beta}}\mathcal{S}^{\alpha}(\underline{x},t;\underline{x}^{o}) \text{ for } (\underline{x},t)\in E'_{\underline{x}^{o}}\times(-\infty,\infty).(2.31)^{11}$$

A physical interpretation of $\mathcal{S}^{\alpha\beta}$ is easily established. Indeed, (2.31) implies

$$\mathcal{S}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}) = \lim_{\theta\to 0}\{\tfrac{1}{\theta}[\mathcal{S}^{\alpha}(\underline{x}+\theta\underline{e}_{\beta},t;\underline{x}^{o}) - \mathcal{S}^{\alpha}(\underline{x},t;\underline{x}^{o})]\},(2.32)$$

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[10] For the analogous treatment of the problem of the half-space under a concentrated surface load perpendicular to the boundary, see Lee [14].

[11] Recall (1.27).

which, because of (2.25), is equivalent to

$$\mathcal{f}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}) = \lim_{\theta \to 0} \{\tfrac{1}{\theta}[\mathcal{f}^{\alpha}(\underline{x},t;\underline{x}^{o}-\theta\underline{e}_{\beta}) - \mathcal{f}^{\alpha}(\underline{x},t;\underline{x}^{o})]\}. \quad (2.33)$$

From (2.31) and (2.26) follows

$$\mathcal{f}^{\alpha\beta}(\underline{x},t;\underline{x}^{o}) = \mathcal{f}^{\alpha\beta}(\underline{x}-\underline{x}^{o},t;0). \quad (2.34)$$

We list next the cartesian components of displacement and stress belonging to $\mathcal{f}^{\alpha\beta}(\underline{x},t;0)$, which may be computed from (2.27) with the aid of the defining relation (2.31).

$$\left.\begin{aligned} u_{i}^{\alpha\beta}(\underline{x},t;0) &= \frac{-1}{8\pi x^{3}}\{[2J_{1}(t)+3Q_{1}(t)]\delta_{i\alpha}x_{\beta} \\ &\quad + [2J_{1}(t)-3Q_{1}(t)][\frac{3x_{i}x_{\alpha}x_{\beta}}{x^{2}} - \delta_{\alpha\beta}x_{i} - \delta_{i\beta}x_{\alpha}]\}, \\ \sigma_{ij}^{\alpha\beta}(\underline{x},t;0) &= \frac{3}{8\pi x^{5}}\{[2h(t)-3Q_{2}(t)][\frac{5x_{\alpha}x_{\beta}x_{i}x_{j}}{x^{2}} - \delta_{\beta i}x_{\alpha}x_{j} \\ &\quad - \delta_{\beta j}x_{\alpha}x_{i} - \delta_{\alpha\beta}x_{i}x_{j}] + Q_{2}(t)[\delta_{\alpha i}(3x_{\beta}x_{j}-\delta_{\beta j}x^{2}) \\ &\quad + \delta_{\alpha j}(3x_{i}x_{\beta}-\delta_{\beta i}x^{2}) - \delta_{ij}(3x_{\alpha}x_{\beta}-\delta_{\alpha\beta}x^{2})]\}. \end{aligned}\right\} \quad (2.35)$$

Here J_1, Q_1, Q_2 are again the response functions given by (2.28).

The subsequent theorem is a trivial consequence of (2.31), Theorem 2.2, and the explicit formulas (2.35).

<u>Theorem</u> 2.3 (Properties of the doublet-states). <u>The doublet-state</u> $\mathcal{f}^{\alpha\beta}(\underline{x},t;0)$ <u>has the properties</u>:

(a) $\underline{u}^{\alpha\beta},\underline{\varepsilon}^{\alpha\beta},\underline{\sigma}^{\alpha\beta} \in H^{\infty,1}(E')$ <u>and</u>

$$\mathcal{f}^{\alpha\beta} = [\underline{u}^{\alpha\beta},\underline{\varepsilon}^{\alpha\beta},\underline{\sigma}^{\alpha\beta}] \in \mathcal{V}(E',G_1,G_2);$$

(b) $\int_{\Pi} \underline{s}^{\alpha\beta}(\underline{x},\cdot;0)dA = 0$ <u>on</u> $(-\infty,\infty)$,

where Π is any closed regular surface surrounding the origin and $\underline{s}^{\alpha\beta}$ is the traction vector of $\mathcal{J}^{\alpha\beta}$ on that side of Π which faces the origin;

$$(c) \quad \int_{\Pi} \underline{x} \wedge \underline{s}^{\alpha\beta}(\underline{x},\cdot;0)dA = \delta_{\alpha\beta\gamma}\underline{e}_{\gamma}h \quad \text{on} \quad (-\infty,\infty),$$

where $\delta_{\alpha\beta\gamma}$ designates the components of the usual alternator;

$$(d) \quad \underline{u}^{\alpha\beta}(\underline{x},\cdot;0) = O(x^{-2}), \underline{\sigma}^{\alpha\beta}(\underline{x},\cdot;0) = O(x^{-3}) \text{ as } x \rightarrow 0,$$

uniformly on $[0,T]$ for every $T \in [0,\infty)$.

Properties (a),(b),(c) were to be anticipated intuitively because of the physical meaning attached to the doublet-states by (2.33), and in view of (a),(b),(c) of Theorem 2.2. As is apparent from (b) and (c) in the present theorem, the stress singularity of $\mathcal{J}^{\alpha\beta}(\underline{x},t;0)$ at the origin is statically equivalent to a couple or to null depending on whether $\alpha \neq \beta$ or $\alpha = \beta$. For this reason we call $\mathcal{J}^{\alpha\beta}(\underline{x},t;\underline{x}^{o})$ a doublet-state with moment or a doublet-state without moment, according as $\alpha \neq \beta$ or $\alpha = \beta$.

As in elasticity theory, it is expedient to introduce two particular linear combinations of doublet-states. Thus we designate by

$$\mathcal{J}^{o}(\underline{x},t;\underline{x}^{o}) = [\underline{u}^{o}(\underline{x},t;\underline{x}^{o}),\underline{\varepsilon}^{o}(\underline{x},t;\underline{x}^{o}),\underline{\sigma}^{o}(\underline{x},t;\underline{x}^{o})] \quad (2.36)$$

the state defined through

$$\mathcal{J}^{o}(\underline{x},t;\underline{x}^{o}) = \mathcal{J}^{\alpha\alpha}(\underline{x},t;\underline{x}^{o}) \quad (2.37)$$ [12]

[12] The summation convention is henceforth understood to apply also to repeated superscripts and to repeated indices that appear once as a subscript and once as a superscript, provided these indices have the range (1,2,3).

and call it the state appropriate to a (viscoelastic) center of compression at $\underline{x}^{o}$. Further, we denote by

$$\bar{\mathcal{J}}^{\alpha}(\underline{x},t;\underline{x}^{o}) = [\bar{\underline{u}}^{\alpha}(\underline{x},t;\underline{x}^{o}),\bar{\underline{\varepsilon}}^{\alpha}(\underline{x},t;\underline{x}^{o}),\bar{\underline{\sigma}}^{\alpha}(\underline{x},t;\underline{x}^{o})] \qquad (2.38)$$

the state defined through

$$\bar{\mathcal{J}}^{\alpha}(\underline{x},t;\underline{x}^{o}) = \frac{1}{2}\,\delta_{\alpha\beta\gamma}\mathcal{J}^{\beta\gamma}(\underline{x},t;\underline{x}^{o}), \qquad (2.39)$$

which we address as the state appropriate to a (viscoelastic) center of rotation in the x_α-direction at $\underline{x}^{o}$.

From (2.37), (2.39), (2.35) follow

$$\left.\begin{aligned} u_i^{o}\,(\underline{x},t;0) &= \frac{-3}{4\pi x^3}\,x_i Q_1(t)\ , \\ \sigma_{ij}^{o}(\underline{x},t;0) &= \frac{3}{4\pi x^5}\,(3x_i x_j-\delta_{ij}x^2)Q_2(t)\,, \end{aligned}\right\}(2.40)$$

$$\left.\begin{aligned} \bar{u}_i^{\alpha}\,(\underline{x},t;0) &= \frac{-1}{4\pi x^3}\,\delta_{\alpha i\beta}x_\beta J_1(t)\ , \\ \bar{\sigma}_{ij}^{\alpha}(\underline{x},t;0) &= \frac{-3}{8\pi x^5}\,(\delta_{\alpha\beta i}x_\beta x_j+\delta_{\alpha\beta j}x_\beta x_i)h(t)\ . \end{aligned}\right\}(2.41)$$

The implications of Theorem 2.3, as far as the properties of the states appropriate to a center of compression and a center of rotation are concerned, are immediate. In particular $\mathcal{J}^{o}(\underline{x},t;0)$ has a self-equilibrated singularity at the origin, whereas the singularity inherent in $\bar{\mathcal{J}}^{\alpha}(\underline{x},t;0)$ is statically equivalent to a couple whose axis is the x_α-axis. Specifically,

$$\int_{\Pi}\bar{\underline{S}}^{\alpha}(\underline{x},\cdot;0)dA = 0\ ,\ \int_{\Pi}\underline{x}\wedge\bar{\underline{S}}^{\alpha}(\underline{x},\cdot;0) = \underline{e}_{\alpha}h \text{ on } (-\infty,\infty). \qquad (2.42)$$

Through successive space-differentiations of the Kelvin-state $\mathcal{J}^{\alpha}(\underline{x},t;\underline{x}^{o})$ one may evidently generate an infinite aggregate of viscoelastic states that are regular on $E'_{\underline{x}^{o}}\times(-\infty,\infty)$ and possess singularities at $\underline{x}^{o}$ of progressively higher order.

3. Green's states. Integral representations for the solution of the fundamental boundary-value problems.

As a prerequisite for the treatment of the main subject of this section we require two lemmas concerning certain integral properties of the viscoelastic Kelvin- and doublet-states discussed in the preceding section. To shorten the formulation of these lemmas, and for future convenience, we make the agreement that $\Sigma_\rho(\underline{x}^o)$ henceforth denotes a spherical surface of radius ρ, centered at the point with the position vector $\underline{x}^o$. Further, $\Omega_\rho(\underline{x}^o)$, as before, denotes a spherical neighborhood of $\underline{x}^o$, with the radius ρ. Finally, we shall write Σ_ρ and Ω_ρ in place of $\Sigma_\rho(\underline{x}^o)$ and $\Omega_\rho(\underline{x}^o)$, when $\underline{x}^o = 0$.

Lemma 3.1. *Let $\mathcal{R}$ be a neighborhood of a point $\underline{x}^o \in E$. Let*

$$\mathcal{J} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}] \in \mathcal{V}(\mathcal{R}, G_1, G_2, \underline{F})$$

and suppose $\mathcal{J}^\alpha(\underline{x}, t; \underline{x}^o)$ is a normalized viscoelastic Kelvin-state. Then for fixed $\underline{x}^o$ and each $t \in (-\infty, \infty)$:

$$\text{(a)} \quad \lim_{\rho \to 0} \int_{\Sigma_\rho(\underline{x}^o)} [\underline{S} * d\underline{u}^\alpha](\underline{x}, t)\, dA = 0, \tag{3.1}$$

$$\text{(b)} \quad \lim_{\rho \to 0} \int_{\Sigma_\rho(\underline{x}^o)} [\underline{S}^\alpha * d\underline{u}](\underline{x}, t)\, dA = u_\alpha(\underline{x}^o, t), \tag{3.2}$$

provided $\underline{S}$ and $\underline{S}^\alpha$ are the respective traction vectors of $\mathcal{J}$ and $\mathcal{J}^\alpha$ on that side of $\Sigma_\rho(\underline{x}^o)$ which faces $\underline{x}^o$.

Proof. Without loss in generality assume $\underline{x}^o = 0$. Also, since $\mathcal{J}(\underline{x}, t)$ and $\mathcal{J}^\alpha(\underline{x}, t; 0)$ vanish for all $(\underline{x}, t) \in \mathcal{R} \times (-\infty, 0)$, it

suffices to consider $t \geq 0$. Thus choose $t \in [0,\infty)$ and hold it fixed throughout the remainder of the argument. Further, let $\rho_0 > 0$ be such that $\Sigma_{\rho_0} \subset \mathcal{R}$.

With a view toward proving (a), define $I_1^\alpha(\rho)$ for all $\rho \in (0,\rho_0]$ by means of

$$I_1^\alpha(\rho) = \int_{\Sigma_\rho} [\underline{S} * d\underline{u}^\alpha](\underline{x},t)\,dA, \qquad (3.3)$$

whence and from (b) in Theorem 1.1,

$$|I_1^\alpha(\rho)| \leq \int_{\Sigma_\rho} |[\underline{u}^\alpha * d\underline{S}](\underline{x},t)|\,dA. \qquad (3.4)$$

Now note that

$$\psi \in H^1 \Longrightarrow \mathcal{V}_\psi(-\infty,t) \leq |\psi(0)| + t \max_{[0,t]} |\dot{\psi}| \;, \qquad (3.5)$$

where $\mathcal{V}_\psi(-\infty,t)$ is the total variation of ψ on $(-\infty,t]$. From (3.4), (3.5), in view of (1.15), (1.8), and the usual estimates of Stieltjes and Riemann integrals, follows

$$|I_1^\alpha(\rho)| \leq [\max_{\Sigma_\rho\times[0,t]} |\underline{u}^\alpha|][\max_{\Sigma_\rho}|\underline{S}(\cdot,0)| + t \max_{\Sigma_\rho\times[0,t]} |\dot{\underline{S}}|]4\pi\rho^2. \quad (3.6)$$

The leading term within brackets is $O(\rho^{-1})$ as $\rho \to 0$ because of (d) in Theorem 2.2, whereas the second term within brackets is uniformly bounded for all $\rho \in (0,\rho_0]$ by virtue of (1.21) and the regularity of τ_{ij} on $\bar{\Omega}_{\rho_0}\times[0,t]$. Hence (3.6) implies (3.1).

Consider next part (b) and define $I_2^\alpha(\rho)$ for all $\rho \in (0,\rho_0]$ through

$$I_2^\alpha(\rho) = \int_{\Sigma_\rho} [\underline{S}^\alpha * d\underline{u}](\underline{x},t)\,dA. \qquad (3.7)$$

Setting

$$\underline{v}(\underline{x},\tau) = \underline{u}(\underline{x},\tau) - \underline{u}(0,\tau) \tag{3.8}$$

for all $(\underline{x},\tau)\in\mathcal{R}\times(-\infty,\infty)$ and invoking once more (b) in Theorem 1.1, one has

$$|I_2^\alpha(\rho) - u_\alpha(0,t)| \le \left|\int_{\Sigma_\rho} [\underline{v}*d\underline{S}^\alpha](\underline{x},t)dA\right|$$

$$+ \left|[\underline{u}(0,\cdot)*d\int_{\Sigma_\rho} \underline{S}^\alpha(\underline{x},\cdot;0)dA](t) - u_\alpha(0,t)\right| . \tag{3.9}$$

The second term in the right-hand member of (3.9) vanishes by virtue of (b) of Theorem 2.2 and (f) of Theorem 1.1. To estimate the first term proceed as in part (a). In this manner one is led to

$$|I_2^\alpha(\rho) - u_\alpha(0,t)| \le$$

$$[\max_{\Sigma_\rho\times[0,t]} |\underline{v}|][\max_{\Sigma_\rho} |\underline{S}^\alpha(\cdot,0;0)| + t \max_{\Sigma_\rho\times[0,t]} |\dot{\underline{S}}^\alpha|]4\pi\rho^2 . \tag{3.10}$$

The leading term within brackets is $o(1)$ as $\rho \to 0$ because of (3.8) and the continuity of $\underline{u}$ on $\bar{\Omega}_{\rho_0}\times[0,t]$. On the other hand, the second term within brackets is $O(\rho^{-2})$ in view of (1.21) and (2.27). Thus (3.2) follows and the proof of Lemma 3.1 is complete.

<u>Lemma</u> 3.2. <u>Let</u> $\mathcal{R}$ <u>be</u> <u>a</u> <u>neighborhood</u> <u>of</u> <u>a</u> <u>point</u> $\underline{x}^0\in E$. <u>Let</u>

$$\mathcal{J} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]\in\mathcal{V}(\mathcal{R},G_1,G_2,\underline{F})$$

<u>and suppose</u> $\mathcal{J}^{\alpha\beta}(\underline{x},t;\underline{x}^0)$ <u>is a viscoelastic doublet-state corresponding to the relaxation functions</u> G_1,G_2. <u>Then for fixed</u> $\underline{x}^0$ <u>and each</u> $t\in(-\infty,\infty)$:

$$\text{(a)}\quad \lim_{\rho\to 0}\int_{\Sigma_\rho(\underline{x}^0)} [\underline{S}*d\underline{u}^{\alpha\beta}](\underline{x},t)\,dA = \frac{2}{5}\,\varepsilon_{\alpha\beta}(\underline{x}^0,t) + \lambda_{\alpha\beta}(\underline{x}^0,t), \tag{3.11}$$

$$\text{(b)}\quad \lim_{\rho\to 0}\int_{\Sigma_\rho(\underline{x}^0)} [\underline{S}^{\alpha\beta}*d\underline{u}](\underline{x},t)\,dA = u_{\beta,\alpha}(\underline{x}^0,t) - \frac{8}{5}\,\varepsilon_{\alpha\beta}(\underline{x}^0,t) + \lambda_{\alpha\beta}(\underline{x}^0,t), \tag{3.12}$$

<u>where</u>

$$\lambda_{\alpha\beta}(\underline{x}^0,t) = \frac{1}{5}\,\{\delta_{\alpha\beta}\varepsilon_{\gamma\gamma}(\underline{x}^0,t) + 2[(\varepsilon_{\alpha\beta}-2\delta_{\alpha\beta}\varepsilon_{\gamma\gamma})*dQ_2](\underline{x}^0,t)\}\ , \tag{3.13}$$

<u>while</u> $\underline{S}$ <u>and</u> $\underline{S}^{\alpha\beta}$ <u>are the respective traction vectors of</u> $\mathcal{S}$ <u>and</u> $\mathcal{S}^{\alpha\beta}$ <u>on that side of</u> $\Sigma_\rho(\underline{x}^0)$ <u>which faces</u> $\underline{x}^0$.

<u>Proof</u>. As in the proof of Lemma 3.1, assume $\underline{x}^0 = 0$, hold $t\in[0,\infty)$ fixed, and let $\rho_0 > 0$ be such that $\Sigma_{\rho_0} \subset \mathcal{R}$.

Consider first part (a) and define $I_1^{\alpha\beta}(\rho)$ for all $\rho\in(0,\rho_0]$ through

$$I_1^{\alpha\beta}(\rho) = \int_{\Sigma_\rho} [\underline{S}*d\underline{u}^{\alpha\beta}](\underline{x},t)\,dA. \tag{3.14}$$

In view of the present hypotheses, one draws from (3.14) in conjunction with (2.35), (1.21), the divergence theorem, Theorem 1.1, and (1.18), after a straightforward computation, that

$$I_1^{\alpha\beta}(\rho) = I_2^{\alpha\beta}(\rho) + I_3^{\alpha\beta}(\rho), \tag{3.15}$$

where

$$I_2^{\alpha\beta}(\rho) = \frac{1}{8\pi\rho^3}\int_{\Omega_\rho}\Big\{[\sigma_{\alpha\beta}*d(2J_1+3Q_1)](\underline{x},t) + [\sigma_{ij}*d(2J_1-3Q_1)](\underline{x},t)\Big[\frac{3}{\rho^2}(\delta_{ij}x_\alpha x_\beta+\delta_{\alpha j}x_i x_\beta+\delta_{\beta j}x_i x_\alpha) - \delta_{\alpha\beta}\delta_{ij}-\delta_{\alpha j}\delta_{\beta i}\Big]\Big\}\,dV, \tag{3.16}$$

$$I_3^{\alpha\beta}(\rho) = -\frac{1}{8\pi\rho^3}\int_{\Omega_\rho}\{[F_\alpha * d(2J_1+3Q_1)](\underline{x},t)x_\beta +$$

$$+ [F_i * d(2J_1-3Q_1)](\underline{x},t)[\frac{3x_i x_\alpha x_\beta}{\rho^2} - \delta_{\alpha\beta}x_i - \delta_{\beta i}x_\alpha]\}\,dV. \quad (3.17)$$

By Definition 1.3 and (a) in Theorem 1.1, the convolutions entering the integrand in (3.16) are (for fixed t) continuous on $\overline{\Omega}_\rho$, whereas the integrand in (3.17) is $o(1)$ as $x \to 0$ so that $I_3^{\alpha\beta}(\rho) = o(1)$ as $\rho \to 0$. Consequently (3.15), (3.16), (3.17) furnish

$$I_1^{\alpha\beta}(\rho) = \frac{1}{8\pi\rho^3}\int_{\Omega_\rho}\{[\sigma_{\alpha\beta} * d(2J_1+3Q_1)](0,t)$$

$$+ [\sigma_{ij} * d(2J_1-3Q_1)](0,t)[\frac{3}{\rho^2}(\delta_{ij}x_\alpha x_\beta + \delta_{\alpha j}x_i x_\beta + \delta_{\beta j}x_i x_\alpha)$$

$$- \delta_{\alpha\beta}\delta_{ij} - \delta_{\alpha j}\delta_{\beta i}]\}\,dV + o(1) \text{ as } \rho \to 0. \quad (3.18)$$

Now carry out the space integration in (3.18) and use (d) of Theorem 1.1, as well as the second of (1.18), to obtain

$$I_1^{\alpha\beta}(\rho) = \frac{1}{15}\{6[\sigma_{\alpha\beta} * d(J_1+Q_1)](0,t)$$

$$- \delta_{\alpha\beta}[\sigma_{ii} * d(2J_1-3Q_1)](0,t)\} + o(1) \text{ as } \rho \to 0. \quad (3.19)$$

Conclusion (a) follows from (3.19), (3.14) after an elementary computation involving the use of Theorem 1.1, Theorem 1.2, equations (1.19),(1.20),(2.28), and the definition of $\lambda_{\alpha\beta}$ given by (3.13).

Turn to part (b) and define $I_4^{\alpha\beta}(\rho)$ for all $\rho \in (0,\rho_0]$ through

$$I_4^{\alpha\beta}(\rho) = \int_{\Sigma_\rho} [\underline{S}^{\alpha\beta} * d\underline{u}](\underline{x},t)dA. \tag{3.20}$$

The required limit of $I_4^{\alpha\beta}(\rho)$, as $\rho \to 0$, may be confirmed by an argument which is quite similar to that employed in establishing part (a). In this connection one needs to make use of the fact — implied by (a) in Definition 1.3 — that, for each $(\underline{x},t) \in \overline{\Omega}_\rho \times [0,\infty)$,

$$\underline{u}(\underline{x},t) = \underline{u}(0,t) + \underline{u}_{,j}(0,t)x_j + \underline{U}(\underline{x},t) \tag{3.21}$$

with $\underline{U} \in C^{2,1}(\overline{\Omega}_\rho \times [0,\infty))$ and

$$\underline{U}(\underline{x},\cdot) = O(x^2) \quad \text{as} \quad x \to 0, \tag{3.22}$$

uniformly on $[0,t]$. Further details of the proof may safely be omitted.

Lemma 3.2, by virtue of (2.39), at once yields

<u>Corollary</u> 3.1. <u>Let</u> $\underline{x}^o, \mathcal{R},$ <u>and</u> $\mathcal{J}$ <u>meet the same hypotheses as in Lemma</u> 3.2. <u>Suppose</u> $\overline{\mathcal{J}}^\alpha(\underline{x},t;\underline{x}^o)$ <u>is the state appropriate to a center of rotation</u>. <u>Then for fixed</u> $\underline{x}^o$ <u>and each</u> $t \in (-\infty,\infty)$:

$$\text{(a)} \lim_{\rho \to 0} \int_{\Sigma_\rho(\underline{x}^o)} [\underline{S} * d\overline{\underline{u}}^\alpha](\underline{x},t)dA = 0,$$

$$\text{(b)} \lim_{\rho \to 0} \int_{\Sigma_\rho(\underline{x}^o)} [\overline{\underline{S}}^\alpha * d\underline{u}](\underline{x},t)dA = \omega_\alpha(\underline{x}^o,t),$$

<u>where</u> $\underline{\omega}$ <u>is the rotation field history of</u> $\mathcal{J}$, <u>i.e.</u>

$$\underline{\omega} = \frac{1}{2} \nabla \wedge \underline{u} \quad \text{on} \quad \mathcal{R} \times (-\infty,\infty), \tag{3.23}$$

<u>while</u> $\underline{S}$ <u>and</u> $\overline{\underline{S}}^\alpha$ <u>are the respective traction vectors of</u> $\mathcal{J}$ <u>and</u> $\overline{\mathcal{J}}^\alpha$ <u>on that side of</u> $\Sigma_\rho(\underline{x}^o)$ <u>which faces</u> $\underline{x}^o$.

It is worth mentioning that conclusions (a),(b) in Lemma 3.1 and Corollary 3.1 do not involve the relaxation functions of either the singular or the regular state under consideration. Also, the conclusions in Lemma 3.1 and Corollary 3.1 hold true even if the relaxation functions of the singular state are distinct from those belonging to the regular state.

We are now in a position to deduce the integral representations which constitute the main objective of this section. With a view toward a representation theorem applicable to the _first_ fundamental boundary-value problem, in which the surface displacements are prescribed over the entire boundary for all time, we introduce

Definition 3.1 (Green's states of the first kind). _We call_

$$\hat{\mathcal{S}}^{\alpha}(\underline{\xi},t;\underline{x}) = [\underline{\hat{u}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\hat{\varepsilon}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\hat{\sigma}}^{\alpha}(\underline{\xi},t;\underline{x})],$$
$$\hat{\mathcal{S}}^{\alpha\beta}(\underline{\xi},t;\underline{x}) = [\underline{\hat{u}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\hat{\varepsilon}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\hat{\sigma}}^{\alpha\beta}(\underline{\xi},t;\underline{x})]$$

the Green's states of the first kind for a (regular) region R _and relaxation functions_ G_1,G_2 _if and only if: for all_ $(\underline{\xi},t;\underline{x})\in\bar{R}\times(-\infty,\infty)\times R$ _with_ $\underline{\xi}\neq\underline{x}$,

$$\left.\begin{aligned}\hat{\mathcal{S}}^{\alpha}(\underline{\xi},t;\underline{x}) &= \mathcal{S}^{\alpha}(\underline{\xi},t;\underline{x}) + \tilde{\mathcal{S}}^{\alpha}(\underline{\xi},t;\underline{x}),\\ \hat{\mathcal{S}}^{\alpha\beta}(\underline{\xi},t;\underline{x}) &= -\tfrac{1}{2}[\mathcal{S}^{\alpha\beta}(\underline{\xi},t;\underline{x}) + \mathcal{S}^{\beta\alpha}(\underline{\xi},t;\underline{x})] + \tilde{\mathcal{S}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\end{aligned}\right\}\quad(3.24)$$

where $\mathcal{S}^{\alpha}(\underline{\xi},t;\underline{x})$ _and_ $\mathcal{S}^{\alpha\beta}(\underline{\xi},t;\underline{x})$ _are the normalized Kelvin-state and the doublet-state, corresponding to_ G_1,G_2; _further_

$$\tilde{\mathcal{J}}^{\alpha}(\underline{\xi},t;\underline{x}) = [\underline{\tilde{u}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\tilde{\varepsilon}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\tilde{\sigma}}^{\alpha}(\underline{\xi},t;\underline{x})],$$

$$\tilde{\mathcal{J}}^{\alpha\beta}(\underline{\xi},t;\underline{x}) = [\underline{\tilde{u}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\tilde{\varepsilon}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\tilde{\sigma}}^{\alpha\beta}(\underline{\xi},t;\underline{x})],$$

for each $\underline{x}\in R$, _are states such that_

(a) $\tilde{\mathcal{J}}^{\alpha}(\cdot,\cdot;\underline{x}),\ \tilde{\mathcal{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x})\in\mathcal{V}(\bar{R},G_1,G_2)$,

(b) $\underline{\hat{u}}^{\alpha}(\cdot,\cdot;\underline{x}) = 0,\ \underline{\hat{u}}^{\alpha\beta}(\cdot,\cdot;\underline{x}) = 0$ _on_ $B\times(-\infty,\infty)$.

Observe that requirements (a),(b), because of (3.24) and Theorem 1.4, for fixed $\underline{x}\in R$, uniquely characterize the states $\tilde{\mathcal{J}}^{\alpha}(\cdot,\cdot;\underline{x})$ and $\tilde{\mathcal{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x})$ as the respective solutions of two first boundary-value problems for the region R. Consequently, the Green's states $\hat{\mathcal{J}}^{\alpha}$ and $\hat{\mathcal{J}}^{\alpha\beta}$ are also uniquely determined by the foregoing definition. With reference to this definition we state

Theorem 3.1 (Integral representation of the solution to the first boundary-value problem). _Suppose_

$$\mathcal{J} = [\underline{u},\underline{\varepsilon},\underline{\sigma}]\in\mathcal{V}(\bar{R},G_1,G_2,\underline{F}). \tag{3.25}$$

Let $\hat{\mathcal{J}}^{\alpha}(\underline{\xi},t;\underline{x})$ and $\hat{\mathcal{J}}^{\alpha\beta}(\underline{\xi},t;\underline{x})$ be the Green's states of the first kind for the region R and relaxation functions G_1,G_2. Then, for each $(\underline{x},t)\in R\times(-\infty,\infty)$,

$$u_\alpha(\underline{x},t) = \int_R [\underline{F}*d\underline{\hat{u}}^{\alpha}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} - \int_B [\underline{\hat{S}}^{\alpha}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}\ , \tag{3.26}$$ [13]

$$\varepsilon_{\alpha\beta}(\underline{x},t) = \int_R [\underline{F}*d\underline{\hat{u}}^{\alpha\beta}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} - \int_B [\underline{\hat{S}}^{\alpha\beta}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}. \tag{3.27}$$

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[13] Here and in the sequel we conveniently write $[\underline{\varphi}*d\underline{\psi}](\underline{\xi},t;\underline{x})$ in place of $[\underline{\varphi}(\cdot,\cdot;\underline{x})*d\underline{\psi}(\cdot,\cdot;\underline{x})](\underline{\xi},t)$ if $\underline{\varphi}(\cdot,\cdot;\underline{x}),\underline{\psi}(\cdot,\cdot;\underline{x})$ are suitable functions of position and time. Cf. the notations adopted in (1.9),(1.15).

Proof. Choose $(\underline{x},t)\in R\times(-\infty,\infty)$ and hold $(\underline{x},t)$ fixed throughout the following argument. Let $\rho_o > 0$ be a number such that $\Sigma_{\rho_o}(\underline{x})\subset R$. For each $\rho\in(0,\rho_o]$ denote by R_ρ the regular region $R-\overline{\Omega}_\rho(\underline{x})$ with the boundary $B\cup\Sigma_\rho(\underline{x})$. Bearing in mind Definition 3.1, as well as Theorems 2.2, 2.3, observe that

$$\hat{\mathscr{J}}^{\alpha}(\cdot,\cdot;\underline{x}),\ \hat{\mathscr{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x})\in\mathscr{V}(\overline{R}_\rho,G_1,G_2). \tag{3.28}$$

Next, apply the reciprocal theorem (Theorem 1.5) to the pair of states $\mathscr{J}$, $\hat{\mathscr{J}}^{\alpha}(\cdot,\cdot;\underline{x})$ and to the pair $\mathscr{J}$, $\hat{\mathscr{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x})$ in their common domain of regularity $\overline{R}_\rho\times(-\infty,\infty)$. Because of (b) in Definition 3.1, this yields

$$\int_{\Sigma_\rho(\underline{x})}[\underline{S}*d\hat{\underline{u}}^{\alpha}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}+\int_{R_\rho}[\underline{F}*d\hat{\underline{u}}^{\alpha}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}}=$$
$$\int_{B}[\hat{\underline{S}}^{\alpha}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}+\int_{\Sigma_\rho(\underline{x})}[\hat{\underline{S}}^{\alpha}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}, \tag{3.29}$$

$$\int_{\Sigma_\rho(\underline{x})}[\underline{S}*d\hat{\underline{u}}^{\alpha\beta}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}+\int_{R_\rho}[\underline{F}*d\hat{\underline{u}}^{\alpha\beta}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}}=$$
$$\int_{B}[\hat{\underline{S}}^{\alpha\beta}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}+\int_{\Sigma_\rho(\underline{x})}[\hat{\underline{S}}^{\alpha\beta}*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}. \tag{3.30}$$

Now proceed to the limit as $\rho\to 0$ in (3.29) and (3.30), using Lemma 3.1 and Lemma 3.2, respectively.[14] This confirms (3.26),(3.27) and completes the proof.

[14] Recall from (3.24) that the singular part of $\hat{\mathscr{J}}^{\alpha}(\cdot,\cdot;\underline{x})$ and of $\hat{\mathscr{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x})$ are a Kelvin-state and a linear combination of doublet-states, respectively. On the other hand, the regular parts of the Green's states under consideration trivially fail to contribute to the limits, as $\rho\to 0$, of the surface integrals over $\Sigma_\rho(\underline{x})$ in (3.29),(3.30).

The relevance of Theorem 3.1 to the first boundary-value problem stems from the fact that the right-hand members of (3.26), (3.27) — apart from elements of the Green's states $\hat{\mathcal{J}}^{\alpha}$, $\hat{\mathcal{J}}^{\alpha\beta}$— involve only the body forces and surface displacements of the state $\mathcal{J}$. Integral representations for the stresses belonging to $\mathcal{J}$, are immediately obtainable from (3.27). If $\bar{R}$, in particular, coincides with the entire space E, then $\hat{\mathcal{J}}^{\alpha} = \mathcal{J}^{\alpha}$ on $E\times(-\infty,\infty)\times E$, according to Definition 3.1, and (3.26) reduces to

$$u_{\alpha}(\underline{x},t) = \int_{E} [\underline{F}*d\underline{u}^{\alpha}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} . \tag{3.31}$$

We now further examine the Green's states entering Theorem 3.1 and establish

Theorem 3.2 (Symmetry of the Green's states of the first kind). *Let $\hat{\mathcal{J}}^{\alpha}(\underline{\xi},t;\underline{x})$ and $\hat{\mathcal{J}}^{\alpha\beta}(\underline{\xi},t;\underline{x})$ be the Green's states of the first kind for a region R. Then, for each $(\underline{\xi},t,\underline{x})\in R\times(-\infty,\infty)\times R$ with $\underline{\xi}\neq\underline{x}$,*

$$\hat{u}^{\alpha}_{\beta}(\underline{\xi},t;\underline{x}) = \hat{u}^{\beta}_{\alpha}(\underline{x},t;\underline{\xi}) , \tag{3.32}$$

$$\hat{\varepsilon}^{\alpha\beta}_{\gamma\delta}(\underline{\xi},t;\underline{x}) = \hat{\varepsilon}^{\gamma\delta}_{\alpha\beta}(\underline{x},t;\underline{\xi}). \tag{3.33}$$

Proof. Choose and fix $(\underline{\xi},t,\underline{x})\in R\times(-\infty,\infty)\times R$ with $\underline{\xi}\neq\underline{x}$. Let $\rho_o > 0$ be a number such that $\Sigma_{\rho_o}(\underline{\xi})\subset R$, $\Sigma_{\rho_o}(\underline{x})\subset R$, while $\Sigma_{\rho_o}(\underline{\xi})\cap\Sigma_{\rho_o}(\underline{x})$ is empty. Then, for each $\rho\in(0,\rho_o]$, the region $R_{\rho} = R-\bar{\Omega}_{\rho}(\underline{\xi}) - \bar{\Omega}_{\rho}(\underline{x})$ is regular and, by hypothesis and Definition 3.1,

$$\hat{\jmath}^{\alpha}(\cdot,\cdot;\underline{x}),\ \hat{\jmath}^{\beta}(\cdot,\cdot;\underline{\xi}) \in \mathcal{V}(\bar{R}_{\rho},G_1,G_2), \tag{3.34}$$

$$\hat{\jmath}^{\alpha\beta}(\cdot,\cdot;\underline{x}),\hat{\jmath}^{\gamma\delta}(\cdot,\cdot;\underline{\xi}) \in \mathcal{V}(\bar{R}_{\rho},G_1,G_2). \tag{3.35}$$

To complete the argument, apply Theorem 1.5 to each of the pairs of states appearing in (3.34) and (3.35), respectively, bear in mind (b) in Definition 3.1, and proceed to the limit as $\rho \rightarrow 0$, making use once again of Lemma 3.1 and Lemma 3.2.

We turn now to the <u>second</u> boundary-value problem (surface tractions prescribed over the entire boundary for all time) and adopt

<u>Definition</u> 3.2 (Green's states of the second kind). <u>We call</u>

$$\hat{\jmath}^{\alpha}(\underline{\xi},t;\underline{x}) = [\hat{\underline{u}}^{\alpha}(\underline{\xi},t;\underline{x}),\hat{\underline{\varepsilon}}^{\alpha}(\underline{\xi},t;\underline{x}),\hat{\underline{\sigma}}^{\alpha}(\underline{\xi},t;\underline{x})],$$

$$\hat{\jmath}^{\alpha\beta}(\underline{\xi},t;\underline{x}) = [\hat{\underline{u}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\hat{\underline{\varepsilon}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\hat{\underline{\sigma}}^{\alpha\beta}(\underline{\xi},t;\underline{x})],$$

<u>the Green's states of the second kind for a (regular) region</u> R, <u>relaxation functions</u> G_1,G_2, <u>and — in case</u> R <u>is bounded — for a (fixed point)</u> $\underline{x}^{o}\in R$ <u>if and only if</u>: <u>for all</u> $(\underline{\xi},t,\underline{x})\in \bar{R}\times(-\infty,\infty)\times R$ <u>with</u> $\underline{\xi}\neq\underline{x}$, $\underline{\xi}\neq\underline{x}^{o}$,

$$\left.\begin{aligned}\hat{\jmath}^{\alpha}(\underline{\xi},t;\underline{x}) &= \jmath^{\alpha}(\underline{\xi},t;\underline{x}) + \tilde{\jmath}^{\alpha}(\underline{\xi},t;\underline{x})\\ &\quad + c[-\jmath^{\alpha}(\underline{\xi},t;\underline{x}^{o}) + \delta_{\alpha\beta\gamma}(x_\beta - x^{o}_\beta)\mathcal{T}^{\gamma}(\underline{\xi},t;\underline{x}^{o})],\\ \hat{\jmath}^{\alpha\beta}(\underline{\xi},t;\underline{x}) &= -\tfrac{1}{2}[\jmath^{\alpha\beta}(\underline{\xi},t;\underline{x}) + \jmath^{\beta\alpha}(\underline{\xi},t;\underline{x})] + \tilde{\jmath}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\end{aligned}\right\} \tag{3.36}$$

<u>where</u> $\jmath^{\alpha}(\underline{\xi},t;\underline{x}),\jmath^{\alpha\beta}(\underline{\xi},t;\underline{x})$, <u>and</u> $\mathcal{T}^{\alpha}(\underline{\xi},t;\underline{x})$ <u>are the normalized Kelvin-state, the doublet-state, and the state appropriate to a center of rotation, all corresponding to</u> G_1,G_2, <u>while</u> c=1 <u>when</u> R <u>is bounded and</u> c=0 <u>when</u> R <u>is unbounded</u>; <u>further</u>,

$$\tilde{\mathcal{J}}^{\alpha}(\underline{\xi},t;\underline{x}) = [\underline{\tilde{u}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\tilde{\varepsilon}}^{\alpha}(\underline{\xi},t;\underline{x}),\underline{\tilde{\sigma}}^{\alpha}(\underline{\xi},t;\underline{x})],$$

$$\tilde{\mathcal{J}}^{\alpha\beta}(\underline{\xi},t;\underline{x}) = [\underline{\tilde{u}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\tilde{\varepsilon}}^{\alpha\beta}(\underline{\xi},t;\underline{x}),\underline{\tilde{\sigma}}^{\alpha\beta}(\underline{\xi},t;\underline{x})],$$

for each $\underline{x} \in R$, *are states such that*

(a) $\tilde{\mathcal{J}}^{\alpha}(\cdot,\cdot;\underline{x}),\ \tilde{\mathcal{J}}^{\alpha\beta}(\cdot,\cdot;\underline{x}) \in \mathcal{V}(\bar{R},G_1,G_2)$,

(b) $\underline{\hat{S}}^{\alpha}(\cdot,\cdot;\underline{x}) = 0,\ \underline{\hat{S}}^{\alpha\beta}(\cdot,\cdot;\underline{x}) = 0$ *on* $B\times(-\infty,\infty)$,

and

(c) *when* R *is bounded*,

$$\underline{\tilde{u}}^{\alpha}(\underline{x}^{o},\cdot;\underline{x}) = \underline{\tilde{\omega}}^{\alpha}(\underline{x}^{o},\cdot;\underline{x}) = 0 \quad \text{on} \quad (-\infty,\infty),$$

$$\underline{\tilde{u}}^{\alpha\beta}(\underline{x}^{o},\cdot;\underline{x}) = \underline{\tilde{\omega}}^{\alpha\beta}(\underline{x}^{o},\cdot;\underline{x}) = 0 \quad \text{on} \quad (-\infty,\infty),$$

in which $\underline{\tilde{\omega}}^{\alpha},\underline{\tilde{\omega}}^{\alpha\beta}$ *are the respective rotation vectors of* $\tilde{\mathcal{J}}^{\alpha}$, $\tilde{\mathcal{J}}^{\alpha\beta}$.

Definition 3.2 is, to an extent, analogous to Definition 3.1. The regular parts of the Green's states in Definition 3.1 are uniquely characterized as solutions to *first* boundary-value problems for the region R. In contrast, $\tilde{\mathcal{J}}^{\alpha}(\cdot,\cdot;\underline{x})$ and $\mathcal{J}^{\alpha\beta}(\cdot,\cdot;\underline{x})$, for fixed $\underline{x} \in R$, are in the present instance each the solution of a *second* boundary-value problem for R, as is apparent from (a), (b), and (3.36). According to Theorem 1.4, the solution to such a problem is unique — except for an arbitrary additive (infinitesimal) rigid motion of the entire body, when R is a *bounded* region.[15] Condition (c) in Definition 3.2 serves to eliminate this indeterminacy from $\underline{\tilde{u}}^{\alpha}$ and $\underline{\tilde{u}}^{\alpha\beta}$.

---

[15] If R is *unbounded*, the additive rigid motion is precluded by the regularity condition (c) in Definition 1.3, which requires the displacements to vanish at infinity. Note also that in the *first* boundary-value problem the additive rigid motion is precluded by the boundary conditions, regardless of whether or not R is bounded.

Thus, for a given $\underline{x}^o$, the states $\tilde{\mathfrak{f}}^\alpha, \tilde{\mathfrak{f}}^{\alpha\beta}$ are fully unique and hence, because of (3.36), the same is true of the Green's states $\hat{\mathfrak{f}}^\alpha, \hat{\mathfrak{f}}^{\alpha\beta}$ of the second kind. Further, when R is unbounded (c=0), both $\hat{\mathfrak{f}}^\alpha$ and $\hat{\mathfrak{f}}^{\alpha\beta}$ are entirely independent of $\underline{x}^o$.

If R is bounded (c=1), the singular part of $\hat{\mathfrak{f}}^\alpha$ in (3.36) is considerably more complicated than its counterpart in (3.24) of Definition 3.1 for reasons to be made clear presently. Indeed, a necessary condition that the second boundary-value problem characterizing $\tilde{\mathfrak{f}}^\alpha$ possess a solution when R is finite, is that for each $\underline{x} \in R$,

$$\int_B \underline{\tilde{S}}^\alpha(\underline{\xi},\cdot;\underline{x})\,dA_{\underline{\xi}} = 0, \quad \int_B \underline{\xi} \wedge \underline{\tilde{S}}^\alpha(\underline{\xi},\cdot;\underline{x})\,dA_{\underline{\xi}} = 0 \text{ on } (-\infty,\infty), \tag{3.37}$$

i.e. that the surface tractions governing $\tilde{\mathfrak{f}}^\alpha(\cdot,\cdot;\underline{x})$ be self-equilibrated.[16] The requirement (3.37), in turn, because of (b) and (3.36), implies that the system of singularities involved in the singular part of $\hat{\mathfrak{f}}^\alpha$ must be self-equilibrated. The supplementary singular part of $\hat{\mathfrak{f}}^\alpha$, which carries the multiplier c and whose singularities are located at $\underline{\xi} = \underline{x}^o$, serves the purpose of assuring the self-equilibrance of the complete system of singularities at $\underline{\xi} = \underline{x}$ and $\underline{\xi} = \underline{x}^o$. This claim is readily verified with the aid of (b),(c) in Theorem 2.2 and (2.42).

---

[16] Note from (a) that $\tilde{\mathfrak{f}}^\alpha(\cdot,\cdot;\underline{x})$ must meet the equilibrium equations in the absence of body forces.

Conditions (3.37) are no longer necessary for the existence of $\mathcal{J}^{\alpha}$ when R is unbounded, in which case c=0 and the singular part of $\hat{\mathcal{J}}^{\alpha}$ is merely a normalized Kelvin-state corresponding to a concentrated load at $\underline{\xi} = \underline{x}$. Finally, we observe that the pair of singularities at $\underline{\xi} = \underline{x}$ in the singular part of $\hat{\mathcal{J}}^{\alpha\beta}$ is always self-equilibrated, regardless of whether or not R is finite, as is immediate from the second of (3.36) and (b),(c) in Theorem 2.3. We may now proceed to

<u>Theorem</u> 3.3 (Integral representation of the solution to the second boundary-value problem). <u>Suppose</u>

$$\mathcal{J} = [\underline{u}, \underline{\varepsilon}, \underline{\sigma}] \in \mathcal{V}(\bar{R}, G_1, G_2, \underline{F}) \tag{3.38}$$

<u>and, if</u> R <u>is bounded,</u>

$$\underline{u}(\underline{x}^{o}, \cdot) = \underline{\omega}(\underline{x}^{o}, \cdot) = 0 \quad \underline{\text{on}} \quad (-\infty, \infty), \tag{3.39}$$

<u>where</u> $\underline{\omega}$ <u>is the rotation vector belonging to</u> $\mathcal{J}$, <u>while</u> $\underline{x}^{o} \in R$. <u>Let</u> $\mathcal{J}^{\alpha}(\underline{\xi}, t; \underline{x})$ <u>and</u> $\hat{\mathcal{J}}^{\alpha\beta}(\underline{\xi}, t; \underline{x})$ <u>be the Green's states of the second kind for the region</u> R, <u>the relaxation functions</u> G_1, G_2, <u>and — in case</u> R <u>is bounded — for</u> $\underline{x}^{o}$. <u>Then, for each</u> $(\underline{x}, t) \in R \times (-\infty, \infty)$,

$$u_{\alpha}(\underline{x}, t) = \int_R [\underline{F} * d\underline{\hat{u}}^{\alpha}](\underline{\xi}, t; \underline{x}) dV_{\underline{\xi}} + \int_B [\underline{S} * d\underline{\hat{u}}^{\alpha}](\underline{\xi}, t; \underline{x}) dA_{\underline{\xi}}, \tag{3.40}$$

$$\varepsilon_{\alpha\beta}(\underline{x}, t) = \int_R [\underline{F} * d\underline{\hat{u}}^{\alpha\beta}](\underline{\xi}, t; \underline{x}) dV_{\underline{\xi}} + \int_B [\underline{S} * d\underline{\hat{u}}^{\alpha\beta}](\underline{\xi}, t; \underline{x}) dA_{\underline{\xi}}. \tag{3.41}$$

<u>Proof</u>. Note that if $\mathcal{J}$ satisfies (3.38) with R bounded, it can always be made to meet (3.39) as well by addition to $\underline{u}$ of a rigid displacement history. The indentity (3.41) and, if R is unbounded, also (3.40), may be confirmed by an argument strictly analogous

Conditions (3.37) are no longer necessary for the existence of $\hat{\mathcal{J}}^{\alpha}$ when R is unbounded, in which case c=0 and the singular part of $\hat{\mathcal{J}}^{\alpha}$ is merely a normalized Kelvin-state corresponding to a concentrated load at $\underline{\xi} = \underline{x}$. Finally, we observe that the pair of singularities at $\underline{\xi} = \underline{x}$ in the singular part of $\hat{\mathcal{J}}^{\alpha\beta}$ is always self-equilibrated, regardless of whether or not R is finite, as is immediate from the second of (3.36) and (b),(c) in Theorem 2.3. We may now proceed to

<u>Theorem</u> 3.3 (Integral representation of the solution to the second boundary-value problem). <u>Suppose</u>

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<u>and, if</u> R <u>is bounded,</u>

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<u>where</u> $\underline{\omega}$ <u>is the rotation vector belonging to</u> $\mathcal{J}$, <u>while</u> $\underline{x}^o \in R$. <u>Let</u> $\hat{\mathcal{J}}^{\alpha}(\underline{\xi},t;\underline{x})$ <u>and</u> $\hat{\mathcal{J}}^{\alpha\beta}(\underline{\xi},t;\underline{x})$ <u>be the Green's states of the second kind for the region</u> R, <u>the relaxation functions</u> G_1, G_2, <u>and — in case</u> R <u>is bounded — for</u> $\underline{x}^o$. <u>Then, for each</u> $(\underline{x},t) \in R\times(-\infty,\infty)$,

$$u_{\alpha}(\underline{x},t) = \int_R [\underline{F}*d\hat{\underline{u}}^{\alpha}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} + \int_B [\underline{S}*d\hat{\underline{u}}^{\alpha}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} \; , \tag{3.40}$$

$$\varepsilon_{\alpha\beta}(\underline{x},t) = \int_R [\underline{F}*d\hat{\underline{u}}^{\alpha\beta}](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} + \int_B [\underline{S}*d\hat{\underline{u}}^{\alpha\beta}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} . \tag{3.41}$$

<u>Proof</u>. Note that if $\mathcal{J}$ satisfies (3.38) with R bounded, it can always be made to meet (3.39) as well by addition to $\underline{u}$ of a rigid displacement history. The indentity (3.41) and, if R is unbounded, also (3.40), may be confirmed by an argument strictly analogous

to that employed in the proof Theorem 3.1. We therefore give a detailed derivation merely for (3.40), assuming R to be bounded.

Consider first the special case in which $\underline{x} = \underline{x}^o$ and conclude from (3.36) that then

$$\hat{\mathcal{J}}^\alpha(\cdot,\cdot;\underline{x}^o) = \tilde{\mathcal{J}}^\alpha(\cdot,\cdot;\underline{x}^o) \quad \text{on} \quad \bar{R}\times(-\infty,\infty). \tag{3.42}$$

But (3.42), together with (a),(b),(c) of Definition 3.2 and Theorem 1.4, implies that $\hat{\mathcal{J}}^\alpha(\cdot,\cdot;\underline{x}^o)$ is the null-state. Hence, and in view of the first of (3.39), the identity (3.40) is trivially met when $\underline{x} = \underline{x}^o$.

Next, hold $(\underline{x},t)\in R\times(-\infty,\infty)$ fixed and assume $\underline{x} \neq \underline{x}^o$. Let $\rho_o > 0$ be a number such that $\Sigma_{\rho_o}(\underline{x})\subset R$, $\Sigma_{\rho_o}(\underline{x}^o)\subset R$, while $\Sigma_{\rho_o}(\underline{x})\cap\Sigma_{\rho_o}(\underline{x}^o)$ is empty. Then, for each $\rho\in(0,\rho_o]$, the region $R_\rho = R-\bar{\Omega}_\rho(\underline{x})-\bar{\Omega}_\rho(\underline{x}^o)$ with the boundary $B\cup\Sigma_\rho(\underline{x})\cup\Sigma_\rho(\underline{x}^o)$ is regular and, by hypothesis and Definition 3.2,

$$\hat{\mathcal{J}}^\alpha(\cdot,\cdot;\underline{x})\in\mathcal{V}(\bar{R}_\rho,a_1,a_2). \tag{3.43}$$

Now apply the reciprocal theorem (Theorem 1.5) to the pair of states $\mathcal{J}$ and $\hat{\mathcal{J}}^\alpha(\cdot,\cdot;\underline{x})$ in their common domain of regularity $\bar{R}_\rho\times(-\infty,\infty)$. Because of (b) in Definition 3.2 one thus arrives at

$$\begin{aligned}
&\int_{R_\rho}[\underline{F}*d\hat{\underline{u}}^\alpha](\underline{\xi},t;\underline{x})dV_{\underline{\xi}} + \int_B[\underline{S}*d\hat{\underline{u}}^\alpha](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} + \\
&\int_{\Sigma_\rho(\underline{x})}[\underline{S}*d\hat{\underline{u}}^\alpha](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} + \int_{\Sigma_\rho(\underline{x}^o)}[\underline{S}*d\hat{\underline{u}}^\alpha](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} = \\
&\int_{\Sigma_\rho(\underline{x})}[\hat{\underline{S}}^\alpha*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}} + \int_{\Sigma_\rho(\underline{x}^o)}[\hat{\underline{S}}^\alpha*d\underline{u}](\underline{\xi},t;\underline{x})dA_{\underline{\xi}}\,.
\end{aligned} \tag{3.44}$$

Proceed to the limit as $\rho \to 0$ in (3.44), taking into account the nature and location of the singularities entering the right-hand member in the first of (3.36) and using Lemma 3.1, Corollary 3.1, as well as (3.39). This yields the desired result (3.40).

It is instructive to examine the influence of the choice of the fixed point $\underline{x}^o$ in Theorem 3.3. A change in this choice evidently affects $\underline{u}$ merely within an additive rigid displacement history and therefore leaves $\underline{\varepsilon}$ unaltered. On the other hand, such a change alters the basic structure of $\hat{\underline{u}}^{\alpha}$ (whose singular part depends upon the location of $\underline{x}^o$) but results only in the addition of a rigid displacement to $\hat{\underline{u}}^{\alpha\beta}$, which is easily seen to have no effect on the right-hand member of (3.41).

The Green's state $\hat{\mathcal{S}}^{\alpha\beta}$ of the second kind conforms to the symmetry relation (3.33) in Theorem 3.2, as may be verified by precisely the same scheme used to prove (3.33) originally. In contrast, $\hat{\mathcal{S}}^{\alpha}$ of Definition 3.2 is found not to obey (3.32) because of the asymmetric manner in which $\underline{\xi}$ and $\underline{x}$ enter the supplementary singular part of $\hat{\mathcal{S}}^{\alpha}$ in the first of (3.36).

Theorem 3.3 owes its importance, as far as the second boundary-value problem is concerned, to the fact that the right-hand members of (3.40),(3.41) — apart from elements of the Green's states of the second kind — involve exclusively the body forces and surface tractions appropriate to $\mathcal{S}$. Analogous integral representations for the stresses belonging to $\mathcal{S}$ follow at once from (3.41) with the aid of (1.19),(1.20).

We have so far confined our attention to integral representations appropriate to the first and second fundamental boundary-value problems. An integral-representation theorem for the _mixed_ problem, which contains Theorem 3.1 and Theorem 3.3 as special cases, may be deduced by obvious similar means and with but superficial complications.

Theorem 3.3 may be used as a basis for a mathematically precise and physically meaningful definition of the notion of a _concentrated surface load_. The latter concept may be defined through a limit process applied to a sequence of regular visco-elastic states that corresponds to distributed surface tractions,[17] in analogy to the limit treatment of internal concentrated loads contained in Section 2. We shall, however, not pursue this issue further and shall turn instead to a more basic application of the integral representation for the solution to the second boundary-value problem supplied by Theorem 3.3.

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[17] See [8], Section 7, for a detailed analysis of concentrated surface loads in _elastostatics_.

4. Saint-Venant's principle for viscoelastic solids.

Saint-Venant's principle in the classical equilibrium theory of elastic solids was originally introduced by Saint-Venant [15] in connection with — and with limitation to — the problem of extension, torsion, and flexure of prismatic or cylindrical bodies. The earliest universal statement of the principle is apparently due to Boussinesq [16](p.298), whose formulation has since become traditional. Love [5](p.132), adopting Boussinesq's version, states the principle as follows: "... the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." The far-reaching importance of the principle stems, of course, from the fact that it presumably entitles one to relax the boundary conditions in the second boundary-value problem of elastostatics and to replace the given surface tractions, at least in part, by statically equivalent — but analytically more manageable — loadings.

As was first pointed out by von Mises [6], the conventional statement of Saint-Venant's principle is in need of clarification. Von Mises observed that the sentence cited involves a dual comparison, which is not fully made explicit. Thus, the statement in question asserts that the strains due to self-equilibrated loads are "negligible" at distances which are

large compared to the size of the load-region, the tacit implication being that these strains are small compared to strains produced by loads which are not statically equivalent to null. Von Mises noted further that one cannot in general meaningfully speak of strains "produced" by non-equilibrated loads since the equilibrium problem for given surface tractions and for a <u>bounded</u> region (in the absence of body forces) has no solution unless the tractions acting on the entire boundary conform to equilibrium. He concluded that for a finite body it is necessary to consider the joint effect of the tractions applied to several (at least two) distinct portions of the boundary.

Two additional criticisms of the traditional version of Saint-Venant's principle are equally self-evident. Trivially, the strains arising from loads applied to a finite part of the surface of an <u>unbounded</u> elastic body are arbitrarily <u>small</u> at points sufficiently far removed from the region of load application, regardless of whether or not the loading is equilibrated. On the other hand, it follows from the superposition principle of linear elasticity theory that the strains and stresses at a fixed point of an elastic body, produced by <u>equilibrated</u> surface loads, may be made as <u>large</u> as one pleases by choosing the magnitude of the loads sufficiently large, regardless of the size of the region of load application.

The foregoing observations suggest an interpretation of the conventional statement of Saint-Venant's principle that may

roughly be phrased as follows[18]: Let the loads acting on an elastic body be confined to several distinct portions of its boundary, each lying within a sphere of radius ρ, and suppose the loads remain bounded as $\rho \to 0$. Then the strains at a fixed interior point of the body are of a smaller order of magnitude in ρ, as $\rho \to 0$, when the tractions on each load region are self-equilibrated than when they are not.

That this is the meaning intended by Boussinesq [16] is apparent from his own efforts to justify the principle. With this objective in mind he examined the strains at an interior point of a semi-infinite elastic body that is subjected to a set of concentrated loads applied <u>normal</u> to its plane boundary. Assuming the points of application of the loads to lie within a sphere of radius ρ, he then showed that the order of magnitude of the strains in question is ρ if the resultant force is zero, and ρ^2 when the resultant moment also vanishes.

Von Mises [6] demonstrated with the aid of two examples, which involve <u>tangential</u> as well as normal surface loads, that the traditional version of Saint-Venant's principle (interpreted as above) requires amendment in order to be generally valid. Guided by these counter-examples he conjectured a modified Saint-Venant principle which was later on formulated and proved in [7]. It is this modified principle which we now seek to extend to <u>viscoelastic</u> solids.

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[18] This interpretation is taken from [7]; it is an elaboration of von Mises' [6] earlier interpretation.

In order to avoid an unduly lengthy statement of the theorem to be established we introduce two preliminary definitions that will permit us to phrase the underlying hypotheses concisely. In this connection we recall our previous agreement to the effect that $\Omega_\rho(\underline{x}^o)$ always stands for an open sphere of radius ρ centered at $\underline{x}^o$.

Definition 4.1 (A set of families of contracting load regions). *We say that* Λ_ρ^n $(0 < \rho < \rho_o;\ n=1,2,\ldots N)$ *is a set of* N *families of load regions on the boundary* B *of a (regular) region* R, *which contract to* N *points* $\overset{o}{\underline{\xi}}{}^n \in B$, *if and only if for every* $\rho \in (0,\rho_o)$ *and every* $n (n=1,2,\ldots N)$:

(a) $\Lambda_\rho^n = \overline{\Omega}_\rho(\overset{o}{\underline{\xi}}{}^n) \cap B$ *and this intersection is connected;*

(b) $\Lambda_\rho^m \cap \Lambda_\rho^n = 0$ *for* $m \neq n$ $(m=1,2, \ldots N)$;

(c) $\Lambda_\rho^n \subset \Pi^n$, *where* Π^n *(the "embedding region" of* Λ_ρ^n*)* *is a subset of* B *and the position vector* $\underline{\xi}$ *of* Π^n *admits the parametrization*

$$\underline{\xi} = \underline{f}^n(\underline{\alpha}) \quad \text{for} \quad \underline{\alpha} = (\alpha_1,\alpha_2) \in D^n, \tag{4.1}$$

D^n *being an open, bounded, simply-connected plane region, while*

$$\underline{f}^n \in C^2(D^n),\ \underline{f}^n(0) = \overset{o}{\underline{\xi}}{}^n,\ \underline{f}^n_{|1} \wedge \underline{f}^n_{|2} \neq 0 \quad \text{on} \quad D^n. \qquad (4.2)^{19}$$

A typical member of the set Λ_ρ^n $(0 < \rho < \rho_o;\ n=1,2,\ldots N)$, for fixed ρ and n, is shown in the accompanying figure, in which

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[19] We use the notation $\underline{f}^n_{|p}(\underline{\alpha}) \equiv \frac{\partial}{\partial\alpha_p} \underline{f}^n(\underline{\alpha})$ $(p=1,2)$. No summation with respect to n is intended in (4.2).

$\overset{o}{Q}{}^n$ and Q designate the endpoints of the position vectors $\overset{o}{\underline{\xi}}{}^n$ and $\underline{\xi}$, respectively, whereas $\underline{x}$ is the position vector of a point P in R. Note that according to the first of (4.2) each subregion Π^n of B possesses continuous curvatures. Therefore the points $\overset{o}{\underline{\xi}}{}^n$ (n=1,2,...N) are necessarily distinct regular points of B; moreover, the boundary B — _within each of the_ N _embedding regions_ — is, by implication, required to exhibit a higher degree of smoothness than that automatically assured by the assumption that R is a regular region of space. The second of (4.2) asserts merely that $\overset{o}{\underline{\xi}}{}^n$ is the image under $\underline{f}^n$ of the origin of the parameter-plane ("$\underline{\alpha}$-plane"). Finally, the third of (4.2) is equivalent to the condition that at each point of D^n at least one of the Jacobians of the mapping $\underline{f}^n$ fails to vanish, so that $\underline{f}^n$ defines a regular curvilinear coordinate-net on Π^n.

Definition 4.2 (Associated family of viscoelastic states). _Let_ Λ_ρ^n $(0 < \rho < \rho_o;\ n=1,2,\ldots N)$ _be a set of_ N _families of load regions on the boundary_ B _of a region_ R, _which contract to_ N _points_ $\overset{o}{\underline{\xi}}{}^n \in B$. _We say that_

$$\mathcal{J}(\underline{x},t,\rho) = [\underline{u}(\underline{x},t,\rho),\underline{\varepsilon}(\underline{x},t,\rho),\underline{\sigma}(\underline{x},t,\rho)] \quad (0 < \rho < \rho_o) \tag{4.3}$$

is a family of viscoelastic states on $\bar{R}\times(-\infty,\infty)$ _corresponding to loads on_ Λ_ρ^n _if and only if for each_ $\rho \in (0,\rho_o)$:

(a) $\mathcal{J}(\cdot,\cdot,\rho) \in \mathcal{V}(\bar{R},a_1,a_2)$;

(b) $\underline{S}(\cdot,\cdot,\rho) = 0$ _on_ $(B - \bigcup_{n=1}^{N} \Lambda_\rho^n)\times(-\infty,\infty)$;

(c) _when_ R _is bounded_,

$\overset{o}{Q}{}^n$ and Q designate the endpoints of the position vectors $\overset{o}{\underline{\xi}}{}^n$ and $\underline{\xi}$, respectively, whereas $\underline{x}$ is the position vector of a point P in R. Note that according to the first of (4.2) each subregion Π^n of B possesses continuous curvatures. Therefore the points $\overset{o}{\underline{\xi}}{}^n$ (n=1,2,...N) are necessarily distinct regular points of B; moreover, the boundary B — <u>within each of the</u> N <u>embedding regions</u> — is, by implication, required to exhibit a higher degree of smoothness than that automatically assured by the assumption that R is a regular region of space. The second of (4.2) asserts merely that $\overset{o}{\underline{\xi}}{}^n$ is the image under $\underline{f}^n$ of the origin of the parameter-plane ("$\underline{\alpha}$-plane"). Finally, the third of (4.2) is equivalent to the condition that at each point of D^n at least one of the Jacobians of the mapping $\underline{f}^n$ fails to vanish, so that $\underline{f}^n$ defines a regular curvilinear coordinate-net on Π^n.

<u>Definition</u> 4.2 (Associated family of viscoelastic states). <u>Let</u> Λ_ρ^n $(0 < \rho < \rho_o;\ n=1,2,\dots N)$ <u>be a set of</u> N <u>families of load regions on the boundary</u> B <u>of a region</u> R, <u>which contract to</u> N <u>points</u> $\overset{o}{\underline{\xi}}{}^n \in B$. <u>We say that</u>

$$\mathcal{J}(\underline{x},t,\rho) = [\underline{u}(\underline{x},t,\rho),\underline{\varepsilon}(\underline{x},t,\rho),\underline{\sigma}(\underline{x},t,\rho)] \quad (0 < \rho < \rho_o) \tag{4.3}$$

<u>is a family of viscoelastic states on</u> $\bar{R}\times(-\infty,\infty)$ <u>corresponding to loads on</u> Λ_ρ^n <u>if and only if for each</u> $\rho \in (0,\rho_o)$:

(a) $\mathcal{J}(\cdot,\cdot,\rho) \in \mathcal{V}(\bar{R},G_1,G_2)$;

(b) $\underline{S}(\cdot,\cdot,\rho) = 0$ <u>on</u> $(B - \bigcup_{n=1}^{N} \Lambda_\rho^n)\times(-\infty,\infty)$;

(c) <u>when</u> R <u>is bounded,</u>

$$\underline{u}(\underline{x}^o,\cdot,\rho) = \underline{\omega}(\underline{x}^o,\cdot,\rho) = 0 \quad \underline{\text{on}} \quad (-\infty,\infty),$$

<u>where</u> $\underline{\omega}$ <u>is the rotation vector of</u> $\mathcal{J}$ <u>and</u> $\underline{x}^o \in R$;

(d) $|\underline{S}|$ <u>is uniformly bounded on</u> $B\times(-\infty,t]\times(0,\rho_o)$ <u>for every</u> $t\in[0,\infty)$.

<u>Furthermore we adopt the notation</u>

$$\left.\begin{aligned} \underline{L}^n(t,\rho) &= \int_{\Lambda^n_\rho} \underline{S}(\underline{\xi},t,\rho)dA , \\ \underline{M}^n(t,\rho) &= \int_{\Lambda^n_\rho} \underline{\xi}\wedge\underline{S}(\underline{\xi},t,\rho)dA , \end{aligned}\right\} \quad (4.4)$$

<u>whence</u> $\underline{L}^n(t,\rho)$ <u>and</u> $\underline{M}^n(t,\rho)$ <u>stand for the resultant force and the resultant moment about the origin, of the tractions</u> $\underline{S}(\cdot,t,\rho)$ <u>acting on</u> Λ^n_ρ .

Assumption (b) requires the surface tractions $\underline{S}(\cdot,t,\rho)$ of the state $\mathcal{J}(\cdot,t,\rho)$ to vanish on the complement with respect to B of the union of the load regions Λ^n_ρ $(n=1,2,\ldots N)$, for all time and every $\rho\in(0,\rho_o)$. It is worth noting that no regularity restrictions other than (d) are placed on the family of states $\mathcal{J}$ as far as its dependence upon the parameter ρ is concerned. Indeed, but for notational complications, it would have been equally adequate for our purposes to introduce a set of <u>sequences</u> of contracting load regions and an associated <u>sequence</u> of viscoelastic states corresponding to loads on these subregions of the boundary. We may now proceed to formulate

Theorem 4.1 (A Saint-Venant principle for viscoelastic solids). Let Λ_ρ^n $(0 < \rho < \rho_o;\ n=1,2,\ldots N)$ be a set of N families of load regions on the boundary B of a (regular) region R, which contract to N points $\overset{o}{\underline{\xi}}{}^n \in B$, and assume $N \geq 2$ if R is bounded. Let $\mathscr{I}(\underline{x},t,\rho)$ $(0 < \rho < \rho_o)$ be a family of viscoelastic states on $\bar{R}\times(-\infty,\infty)$ corresponding to loads on Λ_ρ^n. Let $\underline{x}\in R$ and $t\in(-\infty,\infty)$. Further, assume the existence of the Green's states of the second kind for the region R (Definition 3.2). Then, uniformly on $(-\infty,t]$,

$$\underline{u}(\underline{x},\cdot,\rho) = O(\rho^\delta),\ \underline{\varepsilon}(\underline{x},\cdot,\rho) = O(\rho^\delta),\ \underline{\sigma}(\underline{x},\cdot,\rho) = O(\rho^\delta) \text{ as } \rho \to 0, \tag{4.5}$$

where $\delta=2$. Moreover:

(a) $\delta = 3$ if

$$\underline{L}^n = 0 \text{ on } (-\infty,t]\times(0,\rho_o),\ (n=1,2,\ldots N); \tag{4.6}$$

(b) $\delta = 4$ if

$$\left.\begin{aligned} &\underline{L}^n = 0\ ,\ \int_{\Lambda_\rho^n} \xi_1 \underline{S}(\underline{\xi},\cdot,\cdot)dA = 0 \\ &\text{on } (-\infty,t]\times(0,\rho_o),\ (n=1,2,\ldots N); \end{aligned}\right\} \tag{4.7}$$

(c) $\delta = 4$ if

$$\underline{L}^n = 0,\ \underline{M}^n = 0 \text{ on } (-\infty,t]\times(0,\rho_o),\ (n=1,2,\ldots N), \tag{4.8}$$

provided

$$\underline{S}(\underline{\xi},\tau,\rho) = \Phi^n(\underline{\xi},\tau,\rho)\underline{k}^n(\tau) \quad \text{(no sum)} \tag{4.9}$$

for all $(\underline{\xi},\tau,\rho)\in\Lambda_\rho^n\times(-\infty,t]\times(0,\rho_o)$ and $n=1,2,\ldots N$. Here Φ^n is

scalar-valued, while $\underline{k}^n$ *is continuous with* $\underline{k}^n(\tau)$ *a unit vector such that*

$$\underline{k}^n \cdot \overset{\circ}{\underline{v}}{}^n \neq 0 \quad \text{on} \quad (-\infty, t] \; (n=1,2,\ldots N), \tag{4.10}$$

$\overset{\circ}{\underline{v}}{}^n$ *being the unit normal of* B *at* $\overset{\circ}{\underline{\xi}}{}^n$.

Proof. By hypothesis, (a) in Definition 4.2, and Definition 1.3, $\mathcal{S}(\underline{x},t,\rho)$ is the null state for each $(\underline{x},t,\rho) \in R\times(-\infty,0)\times(0,\rho_o)$; accordingly the conclusion is trivial when $-\infty < t < 0$. Thus choose $(\underline{x},t) \in R\times[0,\infty)$ and hold $(\underline{x},t)$ fixed for the remainder of the argument.

From (a),(b),(c) in Definition 4.2, (b) in Definition 4.1, Theorem 3.3, and (1.19),(1.20) follows, for all $(\tau,\rho) \in (-\infty,\infty)\times(0,\rho_o)$,

$$\left.\begin{aligned} u_i(\underline{x},\tau,\rho) &= \sum_{n=1}^{N} u_i^n(\underline{x},\tau,\rho), \\ \varepsilon_{ij}(\underline{x},\tau,\rho) &= \sum_{n=1}^{N} \varepsilon_{ij}^n(\underline{x},\tau,\rho), \\ \sigma_{ij}(\underline{x},\tau,\rho) &= \sum_{n=1}^{N} \sigma_{ij}^n(\underline{x},\tau,\rho) \end{aligned}\right\} \tag{4.11}$$

where, for each n (n=1,2,...N),

$$\left.\begin{aligned} u_i^n(\underline{x},\tau,\rho) &= \int_{\Lambda_\rho^n} [\underline{S}*d\hat{\underline{u}}^i](\underline{\xi},\tau;\underline{x},\rho)\,dA_{\underline{\xi}} \\ \varepsilon_{ij}^n(\underline{x},\tau,\rho) &= \int_{\Lambda_\rho^n} [\underline{S}*d\hat{\underline{u}}^{ij}](\underline{\xi},\tau;\underline{x},\rho)\,dA_{\underline{\xi}}\,, \\ \sigma_{ij}^n(\underline{x},\tau,\rho) &= [\varepsilon_{ij}^n * dG_1](\underline{x},\tau,\rho) \\ &\quad + \tfrac{1}{3}\,\delta_{ij}[\varepsilon_{kk}^n * d(G_2-G_1)](\underline{x},\tau,\rho). \end{aligned}\right\} \tag{4.12}$$

[20] We write $[\underline{S}*d\hat{\underline{u}}^i](\underline{\xi},t;\underline{x},\rho)$ to denote the convolution-value $[\underline{S}(\cdot,\cdot,\rho)*d\hat{\underline{u}}^i(\cdot,\cdot;\underline{x})](\underline{\xi},t)$, etc. See also (1.9),(1.15), as well as Footnote No. 13.

Here $\underline{\hat{u}}^{i}(\underline{\xi},\tau;\underline{x})$ and $\underline{\hat{u}}^{ij}(\underline{\xi},\tau;\underline{x})$ are the displacement field histories appropriate to the Green's states of the second kind for the region R, the relaxation functions G_1, G_2, and — in case R is bounded — for $\underline{x}^{o}$ (See Definitions 3.2, 4.2).

In view of (4.11),(4.12) it is natural to call $u_i^n, \varepsilon_{ij}^n$, and σ_{ij}^n (n=1,2,...N) the displacement, strain, and stress contributions arising from the loading on the n-th family of load regions. We mention parenthetically, however, that when R is <u>bounded</u> the "contribution-state" $\mathcal{S}^n = [\underline{u}^n, \underline{\varepsilon}^n, \underline{\sigma}^n]$ (n=1,2,...N) possesses an independent <u>physical</u> significance as the solution of a second boundary-value problem if and only if the tractions on each individual family of load regions are permanently self-equilibrated, i.e. $\underline{L}^n = \underline{M}^n = 0$ on $(-\infty,\infty)\times(0,\rho_o)$ for n=1,2,...N. In this case, or when R is unbounded, one has for each $\rho \in (0,\rho_o)$ and each n (n=1,2,...N),

$$\mathcal{S}^n(\cdot,\cdot,\rho) \in \mathcal{V}(\bar{R}, G_1, G_2) \tag{4.13}$$

with

$$\left.\begin{aligned} \underline{S}^n(\cdot,\cdot,\rho) &= \underline{S}(\cdot,\cdot,\rho) \text{ on } \Lambda_\rho^n , \\ \underline{S}^n(\cdot,\cdot,\rho) &= 0 \quad \text{on} \quad B-\Lambda_\rho^n , \end{aligned}\right\} \tag{4.14}$$

which characterize $\mathcal{S}^n$ uniquely. Also, clearly, when R is bounded either the loading on each family of load regions is permanently self-equilibrated or there exist at least two such families for which this is not true.

We now examine a typical displacement contribution $\underline{u}^n(\underline{x},\tau,\rho)$ with a view toward establishing the order of magnitude of $\underline{u}(\underline{x},\tau,\rho)$ as $\rho \to 0$. For this purpose hold n $(n=1,2,\ldots N)$ fixed and observe on the basis of (c) in Definition 4.1 that, for all $(\underline{\xi},\tau)\in \Pi^n\times(-\infty,\infty)$,

$$\hat{\underline{u}}^i(\underline{\xi},\tau;\underline{x}) = \hat{\underline{u}}^i(\underline{f}^n(\underline{\alpha}),\tau;\underline{x}) \equiv \underline{g}^{in}(\underline{\alpha},\tau;\underline{x}),\ \underline{\alpha}=(\alpha_1,\alpha_2), \tag{4.15}$$

where the functions $\underline{g}^{in}(\cdot,\cdot;\underline{x})$ are defined on $D^n\times(-\infty,\infty)$ and evidently

$$\underline{g}^{in}(\cdot,\cdot;\underline{x}) = 0 \quad \text{on} \quad D^n\times(-\infty,0). \tag{4.16}$$[21]

As is clear from the behavior of $\hat{\underline{u}}^i(\cdot,\cdot;\underline{x})$ on $B\times[0,\infty)$ implied by Definition 3.2 and the smoothness of $\underline{f}^n$ stipulated in the first of (4.2), the functions

$$\dot{\underline{g}}^{in}(\underline{\alpha},\tau;\underline{x}) \equiv \frac{\partial}{\partial\tau}\,\underline{g}^{in}(\underline{\alpha},\tau;\underline{x}) \tag{4.17}$$

exist and are continuous for all $(\underline{\alpha},\tau)\in D^n\times[0,\infty)$; furthermore, throughout this domain $\underline{g}^{in}(\underline{\alpha},\tau;\underline{x})$ and $\dot{\underline{g}}^{in}(\underline{\alpha},\tau;\underline{x})$ are twice continuously differentiable with respect to α_1,α_2.

In view of the preceding observations, $\underline{g}^{in}(\cdot,\tau;\underline{x})$ for each $\tau\in(-\infty,\infty)$ and all $\underline{\alpha}\in D^n$ admits the Taylor expansion

$$\underline{g}^{in}(\underline{\alpha},\tau;\underline{x}) = \overset{o}{\underline{g}}{}^{in}(\tau;\underline{x}) + \overset{o}{\underline{g}}{}^{in}_{|p}(\tau;\underline{x})\alpha_p + \underline{\theta}^{in}(\underline{\alpha},\tau;\underline{x}) \tag{4.18}$$[22]

where we have used the notation

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[21] Recall that $\underline{x}\in R$ is fixed.

[22] Here and in the sequel summation with respect to p $(p=1,2)$ is implied when p is a repeated index.

$$\left.\begin{aligned} &\overset{o}{\underline{g}}{}^{in}(\tau;\underline{x}) \equiv \underline{g}^{in}(0,\tau;\underline{x}), \\ &\overset{o}{\underline{g}}{}^{in}_{|p}(\tau;\underline{x}) \equiv \frac{\partial}{\partial \alpha_p} \underline{g}^{in}(\underline{\alpha},\tau;\underline{x})\Big|_{(\alpha_1=\alpha_2=0)} \quad (p=1,2). \end{aligned}\right\} \quad (4.19)$$

The remainder $\underline{\theta}^{in}(\underline{\alpha},\tau;\underline{x})$ in (4.18) evidently possesses the same degree of smoothness for all $(\underline{\alpha},\tau) \in D^n \times [0,\infty)$ as does $\underline{g}^{in}(\underline{\alpha},\tau;\underline{x})$. Furthermore,

$$\underline{\theta}^{in}(\cdot,\cdot;\underline{x}) = 0 \quad \text{on} \quad D^n \times (-\infty,0), \qquad (4.20)$$

whereas

$$\underline{\theta}^{in}(\underline{\alpha},\cdot;\underline{x}) = O(\alpha^2),\ \dot{\underline{\theta}}^{in}(\underline{\alpha},\cdot;\underline{x}) = O(\alpha^2) \text{ as } \alpha \equiv |\underline{\alpha}| \rightarrow 0, \qquad (4.21)$$

uniformly on $(-\infty,t]$.

Before continuing the argument we note from (4.1),(4.2) that the mapping $\underline{f}^n$ is one-to-one in a neighborhood of the origin of the parameter-plane. Indeed, (4.1),(4.2) together with (a) in Definition 4.1 imply that there exists a number $\rho_1 (0 < \rho_1 < \rho_o)$ and a function $\underline{\varphi}^n$ mapping $\Lambda^n_{\rho_1}$ onto a neighborhood of the origin of the $\underline{\alpha}$-plane, i.e.

$$\underline{\alpha} = \underline{\varphi}^n(\underline{\xi}), \qquad (4.22)$$

where $\underline{\varphi}^n$ is independent of one of the components of $\underline{\xi}$ and is continuously differentiable with respect to the remaining two. In order to avoid cumbersome notation and since n is being held fast, we shall hereafter write $\underline{\alpha}(\underline{\xi})$ in place of $\underline{\varphi}^n(\underline{\xi})$. It follows from the regularity of the inverse mapping under consideration that

$$\max_{\underline{\xi} \in \Lambda_\rho^n} |\underline{\alpha}(\underline{\xi})| = O(\rho), \quad \int_{\Lambda_\rho^n} dA = O(\rho^2) \quad \text{as } \rho \to 0. \tag{4.23}$$

Now substitute from (4.15) into the first of (4.12) and use (4.18),(4.22). After a brief computation involving permissible reversals in the order of the processes of convolution and surface integration one thus obtains, for all $(\tau,\rho) \in (-\infty,t]\times(0,\rho_1)$,

$$u_i^n(\underline{x},\tau,\rho) = I_1^{in}(\underline{x},\tau,\rho) + I_2^{in}(\underline{x},\tau,\rho) + I_3^{in}(\underline{x},\tau,\rho), \tag{4.24}$$

where

$$\left.\begin{aligned} I_1^{in}(\underline{x},\tau,\rho) &= [\int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\cdot,\rho)dA * d\overset{o}{\underline{g}}{}^{in}(\cdot;\underline{x})](\tau), \\ I_2^{in}(\underline{x},\tau,\rho) &= [\int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\cdot,\rho)\alpha_p(\underline{\xi})dA * d\overset{o}{\underline{g}}{}^{in}_{|p}(\cdot;\underline{x})](\tau), \\ I_3^{in}(\underline{x},\tau,\rho) &= \int_{\Lambda_\rho^n} [\underline{S}(\underline{\xi},\cdot,\rho) * d\underline{\theta}^{in}(\underline{\alpha}(\underline{\xi}),\cdot;\underline{x})](\tau)dA_{\underline{\xi}}. \end{aligned}\right\} \tag{4.25}$$

Also, by (4.16),(4.19) and (4.20),(4.25),

$$I_k^{in}(\underline{x},\cdot,\cdot) = 0 \quad \text{on} \quad (-\infty,0)\times(0,\rho_1). \tag{4.26}$$

Our next task consists in estimating the order of magnitude of $I_k^{in}(\underline{x},\tau,\rho)$ as $\rho \to 0$. Consider first I_3^{in} in (4.25) and observe with the aid of (1.15) and (3.5) that, for all $(\tau,\rho) \in [0,t]\times(0,\rho_1)$,

$$|I_3^{in}(\underline{x},\tau,\rho)| \leq$$

$$[\max_{\Lambda_\rho^n \times [0,t]} |\underline{S}(\cdot,\cdot,\rho)|][\max_{\underline{\xi} \in \Lambda_\rho^n} |\underline{\Theta}^{in}(\underline{\alpha}(\underline{\xi}),0;\underline{x})|$$

$$+ t \max_{(\underline{\xi},\tau) \in \Lambda_\rho^n \times [0,t]} |\dot{\underline{\Theta}}^{in}(\underline{\alpha}(\underline{\xi}),\tau;\underline{x})|] \int_{\Lambda_\rho^n} dA \,. \tag{4.27}$$

By (4.27), (d) in Definition 4.2, (4.23), and (4.21), there exist constants $\rho_2 (0 < \rho_2 < \rho_1)$ and C such that, for all $(\tau,\rho) \in [0,t] \times (0,\rho_2)$,

$$|I_3^{in}(\underline{x},\tau,\rho)| < C\rho^4 + Ct\rho^4. \tag{4.28}$$

This conclusion, together with (4.26), assures that

$$I_3^{in}(\underline{x},\cdot,\rho) = O(\rho^4) \text{ as } \rho \to 0, \tag{4.29}$$

uniformly on $(-\infty,t]$. Proceeding similarly with the first two of (4.25) — bearing in mind (4.17),(4.19), and the continuity of $\dot{\underline{g}}^{in}(0,\cdot;\underline{x})$ and $\dot{\underline{g}}^{in}_{|p}(0,\cdot;\underline{x})$ on $[0,t]$ — one arrives at

$$\left.\begin{aligned} I_1^{in}(\underline{x},\cdot,\rho) &= O(\rho^2) \text{ as } \rho \to 0 \,, \\ I_2^{in}(\underline{x},\cdot,\rho) &= O(\rho^3) \text{ as } \rho \to 0 \,, \end{aligned}\right\} \tag{4.30}$$

uniformly on $(-\infty,t]$. But (4.29),(4.30), because of (4.24) and the first of (4.11), imply the first of (4.5) with $\delta=2$.

We now turn to the remaining assertions concerning the order of magnitude of the displacements as $\rho \to 0$, which presuppose various restrictions upon the loading beyond those

implied by Definition 4.2. Thus suppose (4.6) holds so that the resultant force of the loading on each family of load regions vanishes up to the instant t. In this case, $I_1^{in}(\underline{x},\cdot,\cdot)$ vanishes on $(-\infty,t]\times(0,\rho_1)$ according to (4.4),(4.25), and hence (4.24), (4.29) imply

$$u_1^n(\underline{x},\cdot,\rho) = I_2^{in}(\underline{x},\cdot,\rho) + O(\rho^4) \text{ as } \rho \to 0, \qquad (4.31)$$

uniformly on $(-\infty,t]$. The conclusion under case (a) is immediate from (4.31) and the second of (4.30) in conjunction with the first of (4.11).

Next consider case (b), which is characterized by (4.7). Here (4.6) continues to hold, whence (4.31) remains valid. In addition, the second of (4.7) requires the three first moments (about the coordinate planes) of the tractions on each family of load regions to vanish up to time t.

Equations (4.1),(4.2) insure that $\underline{f}^n$ on D^n admits the Taylor expansion

$$\underline{f}^n(\underline{\alpha}) = \overset{\circ}{\underline{f}}{}^n + \overset{\circ}{\underline{f}}{}^n_{|p}\alpha_p + \underline{\psi}^n(\underline{\alpha}),\ \underline{\alpha} = (\alpha_1,\alpha_2), \qquad (4.32)^{23}$$

where

$$\overset{\circ}{\underline{f}}{}^n \equiv \underline{f}^n(0) = \overset{\circ}{\underline{\xi}}{}^n\ ,\ \overset{\circ}{\underline{f}}{}^n_{|p} \equiv \underline{f}^n_{|p}(0) \quad (p=1,2). \qquad (4.33)$$

The remainder $\underline{\psi}^n \in C^2(D^n)$ and

$$\underline{\psi}^n(\underline{\alpha}) = O(\alpha^2) \text{ as } \alpha \equiv |\underline{\alpha}| \to 0. \qquad (4.34)$$

Since $\underline{\xi} = \underline{f}^n(\underline{\alpha})$ on Π^n, equations (4.32),(4.7), in view of the first of (4.4), at once furnish

[23] Recall Footnotes No. 19,22.

$$\overset{\circ n}{f}_{1|p} \int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA = - \int_{\Lambda_\rho^n} \psi_1^n(\underline{\alpha}(\underline{\xi}))\underline{S}(\underline{\xi},\tau,\rho)dA \tag{4.35}$$

for all $(\tau,\rho) \in (-\infty,t]\times(0,\rho_1)$. On estimating the right-hand member of (4.35) with the aid of (4.34),(4.23), and (d) in Definition 4.2, one infers that

$$\overset{\circ n}{f}_{1|p} \int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA = O(\rho^4) \text{ as } \rho \to 0, \tag{4.36}$$

uniformly for all $\tau \in (-\infty,t]$. Now, (4.36) may be regarded as a system of three (inhomogeneous) linear algebraic equations in the two unknowns

$$\int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA \ (p=1,2).$$

Furthermore, because of the last of (4.2), the coefficient-matrix of this system has the rank two. Hence (4.36) imply

$$\int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\cdot,\rho)\alpha_p(\underline{\xi})dA = O(\rho^4) \text{ as } \rho \to 0 \quad (p=1,2), \tag{4.37}$$

uniformly on $(-\infty,t]$. From (4.37) and the second of (4.25), in turn, follows the estimate

$$I_2^{in}(\underline{x},\cdot,\rho) = O(\rho^4) \text{ as } \rho \to 0, \tag{4.38}$$

uniformly on $(-\infty,t]$. Finally, (4.38) together with (4.31) and the first of (4.11) imply the first of (4.5) with $\delta=4$.

Consider at last case (c). Here the ordinary equilibrium conditions (4.8) are presumed to hold, whence the loading on each family of load regions is self-equilibrated. In addition, as required by (4.9), the tractions on each family Λ_ρ^n (n fixed) — up to time t — are now supposed to form a parallel system at every instant and, according to (4.10), must not be parallel to the tangent plane[24] of B at $\overset{\circ}{\underline{\xi}}{}^n$.

Since (4.8) include the assumption (4.6) underlying case (a), equation (4.31) is satisfied also in case (c). Further, equations (4.8), by virtue of (4.4) and (4.32), yield

$$\overset{\circ}{\underline{f}}{}^n_{|p} \wedge \int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA = -\int_{\Lambda_\rho^n} \underline{\psi}^n(\underline{\alpha}(\underline{\xi}))\wedge \underline{S}(\underline{\xi},\tau,\rho)dA \quad (4.39)$$

for all $(\tau,\rho)\in(-\infty,t]\times(0,\rho_1)$. From (4.39) one draws[25]

$$\overset{\circ}{\underline{f}}{}^n_{|p} \wedge \int_{\Lambda_\rho^n} \underline{S}(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA = O(\rho^4) \text{ as } \rho \to 0, \quad (4.40)$$

uniformly for all $\tau\in(-\infty,t]$. Next, substitute from (4.9) into (4.40) to obtain

$$\overset{\circ}{\underline{f}}{}^n_{|p} \wedge \underline{k}^n(\tau) \int_{\Lambda_\rho^n} \Phi^n(\underline{\xi},\tau,\rho)\alpha_p(\underline{\xi})dA = O(\rho^4) \text{ as } \rho \to 0, \quad (4.41)$$

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24 It follows from the assumed smoothness of the boundary B within each embedding region Π^n that the instantaneous tractions on Λ_ρ^n, for fixed n and sufficiently small ρ, cannot at present be parallel to B <u>anywhere</u> within Λ_ρ^n.

25 Cf. the estimate of the right-hand member of (4.35).

uniformly for all $\tau \in (-\infty, t]$. Scalar multiplication of (4.41) by $\overset{\circ}{\underline{f}}{}^n_{|q}$ (q=1,2) leads to

$$[(\overset{\circ}{\underline{f}}{}^n_{|1} \wedge \overset{\circ}{\underline{f}}{}^n_{|2}) \cdot \underline{k}^n(\tau)] \int_{\Lambda^n_\rho} \Phi^n(\underline{\xi}, \tau, \rho) \alpha_p(\underline{\xi}) dA = O(\rho^4) \text{ as } \rho \to 0 \ (p=1,2), \tag{4.42}$$

uniformly for all $\tau \in (-\infty, t]$. Now note from (4.2) that $\overset{\circ}{\underline{f}}{}^n_{|1} \wedge \overset{\circ}{\underline{f}}{}^n_{|2}$ is a non-zero vector that is normal to the boundary B at $\overset{\circ}{\underline{\xi}}{}^n$. Consequently and by the hypotheses on $\underline{k}^n$, the coefficient of the integral in (4.42) is uniformly bounded away from zero for all $\tau \in (-\infty, t]$, whence — using (4.9) once more —

$$\int_{\Lambda^n_\rho} \underline{S}(\underline{\xi}, \cdot, \rho) \alpha_p(\underline{\xi}) dA = O(\rho^4) \text{ as } \rho \to 0 \ (p=1,2), \tag{4.43}$$

uniformly on $(-\infty, t]$. But (4.43) is identical with (4.37) and thus the first of (4.5) with $\delta=4$ follows in the same manner as in case (b).

This completes the proof as far as the required orders of magnitude of the displacements are concerned. To reach the corresponding conclusions regarding the <u>strains</u> one may proceed through the identical argument, taking the second of (4.11) and the second integral representation in (4.12) as the point of departure. Since the foregoing reasoning in no way depended upon the specific nature of the singularities inherent in the Green's displacement $\underline{\hat{u}}^1$ entering the first of (4.12), the desired conclusions for the strains are immediate from those we have

established already. Finally, the requisite orders of magnitude of the stresses follow trivially from those pertaining to the strains because of the last of (4.11) and the third of (4.12). The proof of Theorem 4.1 is now complete in its entirety.

Observe that the content of Theorem 4.1 is not weakened if the conclusions $\delta=2$, $\delta=3$, $\delta=4$ are replaced by $\delta \geq 2$, $\delta \geq 3$, $\delta \geq 4$, respectively. The essential significance of the first conclusion ($\delta=2$) lies in the fact that the displacements, strains, and stresses (at a fixed interior point of the body) are bound to vanish at least to the order $O(\rho^2)$ as $\rho \to 0$ in the absence of any restrictions upon the loading beyond those implied by Definition 4.2. In particular, this order of magnitude prevails regardless of whether or not the loading on each family of load regions is or is not self-equilibrated.

In case (a), where (4.6) is met, so that the resultant force belonging to each family of load regions vanishes, a reduction of the (maximum) order of magnitude from $O(\rho^2)$ to $O(\rho^3)$ is guaranteed. But (4.6), though sufficient, is clearly not necessary for $\delta=3$. Analogous comments apply to the further reduction from $O(\rho^3)$ to $O(\rho^4)$ assured in case (b) and case (c).

Conditions (4.7) evidently imply the ordinary equilibrium conditions (4.8). The converse is however not true, whence (4.7) represent a stronger restriction upon the loading than do (4.8). On the basis of the traditional statement of Saint-Venant's principle in elastostatics, discussed at the beginning of this

section, one would expect (4.8) by themselves — i.e. in the absence of the additional requirement (4.9),(4.10) that the tractions on each family of load regions be parallel and not tangential to the boundary — to guarantee the reduction to $\delta=4$ from $\delta=3$ in case (a). That this expectation is not borne out by the facts is apparent from von Mises' [6] counter-examples, which refer to the special case of the elastic solid.

The preceding conclusion has a counterpart in the theory of basic singular states dealt with in Section 2. Thus consider the Kelvin-state $\mathscr{J}^{\alpha}(\underline{x},t;0)$ and the doublet-state $\mathscr{J}^{\alpha\beta}(\underline{x},t;0)$. Both of these states are regular for all $(\underline{x},t)\in(E-R)\times(-\infty,\infty)$, where R is an arbitrary (regular) region containing the origin, and both states may be regarded as induced on $(E-R)\times(-\infty,\infty)$ by their respective surface tractions on the boundary Π of E-R. Let $\underline{L}^{\alpha}$, $\underline{M}^{\alpha}$ and $\underline{L}^{\alpha\beta}$, $\underline{M}^{\alpha\beta}$ denote the resultant force and the resultant moment about the origin of the tractions on Π belonging to $\mathscr{J}^{\alpha}$ and $\mathscr{J}^{\alpha\beta}$, respectively. Then, from Theorem 2.2 and Theorem 2.3, on $[0,\infty)$,

$$\left.\begin{aligned}
&\underline{L}^{\alpha} \neq 0\,, \quad \underline{M}^{\alpha} = 0\,,\\
&\underline{L}^{\alpha\beta} = 0\,, \quad \underline{M}^{\alpha\beta} \neq 0 \quad (\alpha\neq\beta),\\
&\underline{L}^{\alpha\beta} = 0\,, \quad \underline{M}^{\alpha\beta} = 0 \quad (\alpha=\beta).
\end{aligned}\right\}\qquad(4.44)$$

Next consider the rate of decay of the corresponding stresses at infinity (i.e. "at distances large compared to the size of the region of load application Π"). An inspection of (2.27),(2.35)

confirms at once that for every $t \in [0,\infty)$, as $x \equiv |\underline{x}| \to \infty$,

$$\left.\begin{aligned} &\underline{\sigma}^{\alpha}(\underline{x},t;0) = O(x^{-2}),\ \underline{\sigma}^{\alpha}(\underline{x},t;0) \neq O(x^{-\delta})\ (\delta > 2), \\ &\underline{\sigma}^{\alpha\beta}(\underline{x},t;0) = O(x^{-3}),\ \underline{\sigma}^{\alpha\beta}(\underline{x},t;0) \neq O(x^{-\delta})\ (\delta > 3),\ (\alpha \neq \beta), \\ &\underline{\sigma}^{\alpha\beta}(\underline{x},t;0) = O(x^{-3}),\ \underline{\sigma}^{\alpha\beta}(\underline{x},t;0) \neq O(x^{-\delta})\ (\delta > 3),\ (\alpha = \beta). \end{aligned}\right\} \quad (4.45)$$

Consequently, whereas the stresses decay more rapidly when the resultant force of the loading on Π vanishes than when this is not the case, the additional vanishing of the resultant moment fails to give rise to a further reduction in the order of magnitude of the stresses as $x \to \infty$.

The Saint-Venant principle contained in Theorem 4.1 may be extended to accommodate <u>concentrated surface loads</u> with the aid of a corresponding generalization[26] of Theorem 3.3. If this is done, one finds that Theorem 4.1 continues to hold true provided the conclusions $\delta=2$, $\delta=3$, $\delta=4$ are replaced by $\delta=0$, $\delta=1$, $\delta=2$, respectively. On the other hand, the extension of the principle to <u>anisotropic</u> viscoelastic materials would require an integral representation for the solution of the second boundary-value problem appropriate to such materials and analogous to that deduced by Fredholm [17] in the linear equilibrium theory of anisotropic elastic solids.

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[26] Cf. the remark on concentrated surface loads at the end of Section 3.

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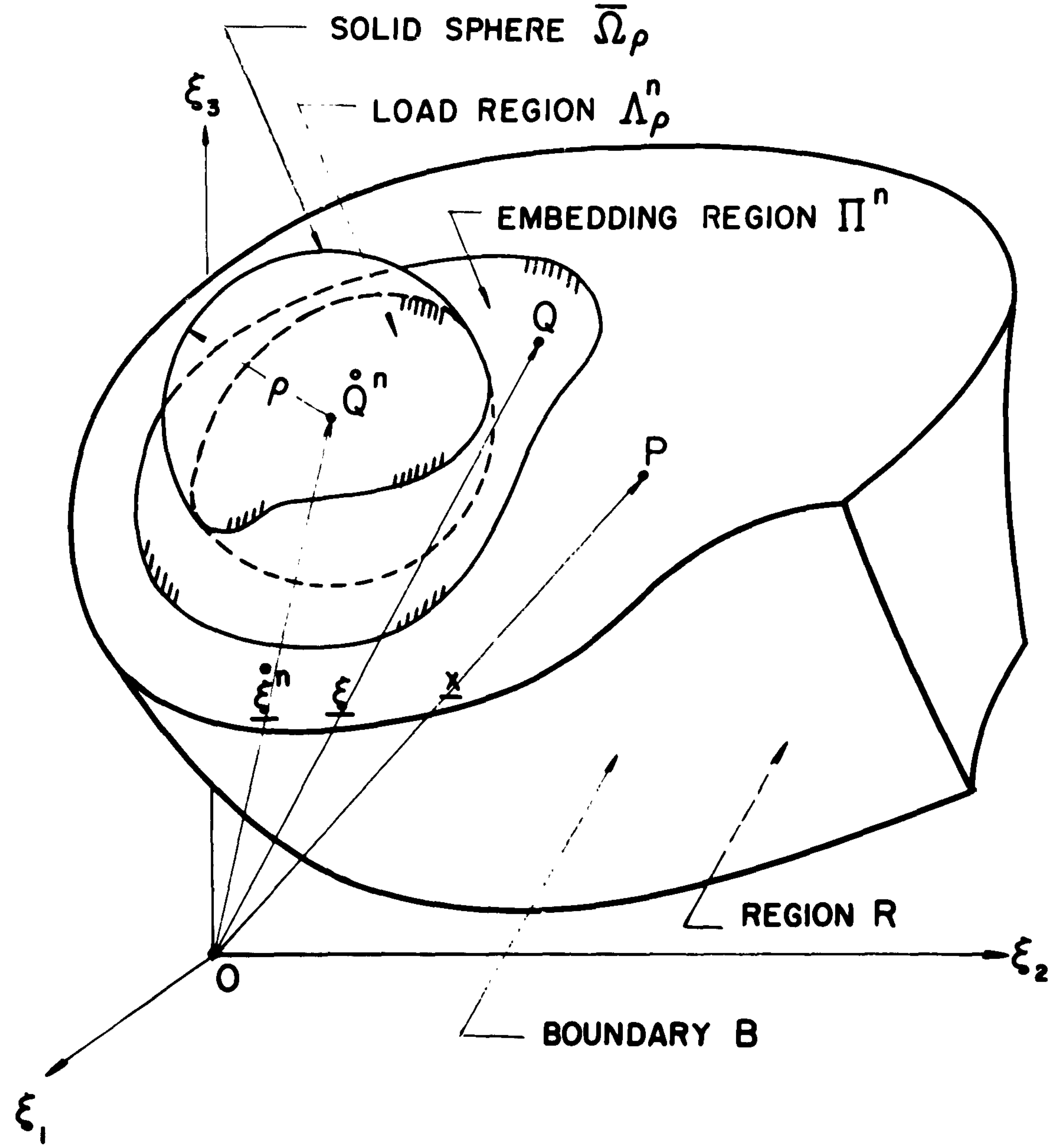
SOLID SPHERE $\overline{\Omega}_\rho$
LOAD REGION Λ_ρ^n
EMBEDDING REGION Π^n
ξ_3
Q
ρ
$\mathring{Q}^n$
P
$\underline{\mathring{\xi}}^n$
$\underline{\xi}$
$\underline{x}$
REGION R
O
ξ_2
BOUNDARY B
ξ_1